

MANIN'S CONJECTURE FOR CERTAIN SPHERICAL THREEFOLDS

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ABSTRACT. We prove Manin's conjecture on the asymptotic behavior of the number of rational points of bounded anticanonical height for a spherical threefold with canonical singularities and two infinite families of spherical threefolds with log terminal singularities. Moreover, we show that one of these families does not satisfy a conjecture of Batyrev and Tschinkel on the leading constant in the asymptotic formula. Our proofs are based on the universal torsor method, using Brion's description of Cox rings of spherical varieties.

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1. INTRODUCTION

1.1. Spherical varieties and Manin's conjecture. Manin's conjecture [FMT89, BM90, Pey95, BT98b, Pey03, Pey16] makes a precise prediction for the asymptotic behavior of the number of rational points of bounded anticanonical height on (almost) Fano varieties over number fields whose set of rational points is Zariski dense.

For a smooth Fano variety over \mathbb{Q} with a Zariski dense set of rational points, one may introduce an anticanonical height function $H: X(\mathbb{Q}) \rightarrow \mathbb{R}_{>0}$ and ask for the asymptotic behavior of the number of rational points of bounded height, as the height bound tends to infinity. The total number might be dominated by points on *accumulating* subvarieties (or, more generally, accumulating *thin subsets*, see [Pey03, §8]), and hence it is more interesting to restrict to their complement U . By [BM90, Conjecture B'], we are lead to the expectation that

$$N_{U,H}(B) := \#\{x \in U(\mathbb{Q}) : H(x) \leq B\} \sim \mathfrak{c} B(\log B)^{\rho-1}$$

as $B \rightarrow \infty$, where ρ is the Picard number of X . A conjecture for the leading constant \mathfrak{c} is given by Peyre in [Pey95]. If X is a singular Fano variety with a *crepant resolution* $\pi: \tilde{X} \rightarrow X$ (i.e., a desingularization with $\pi^*(-K_X) = -K_{\tilde{X}}$), then [BM90, Conjecture C'] and [Pey03, 5.1] tells us that such an asymptotic formula

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should hold with ρ and \mathfrak{c} computed on \tilde{X} . If X has worse singularities, [BM90, Conjecture C'] and [Pey03, 3.6] predict

$$N_{U,H}(B) \sim \mathfrak{c} B^{\mathfrak{a}} (\log B)^{\mathfrak{b}-1},$$

where we may have $\mathfrak{a} > 1$; Batyrev and Tschinkel [BT98b] give a prediction for \mathfrak{c} .

Manin's conjecture has been proved for some classes of varieties and several individual examples. Most of the known cases are proved using either harmonic analysis on adelic points or the universal torsor method combined with various analytic techniques.

Many of them are *spherical varieties*, i.e., normal G -varieties containing a dense B -orbit, where G is a connected reductive group and $B \subseteq G$ is a Borel subgroup. Spherical varieties are a huge class of varieties that admit a combinatorial description by spherical systems (Luna's program [Lun01]) and colored fans (Luna–Vust theory [LV83]) generalizing the combinatorial description of toric varieties.

In particular, harmonic analysis has been used to prove Manin's conjecture for some classes of equivariant compactifications of algebraic groups, for example flag varieties [FMT89], toric varieties [BT98a], horospherical varieties [ST99], and wonderful compactifications of semi-simple groups [GMO08, STBT07]. All these varieties are spherical varieties; more precisely, flag varieties and toric varieties are special cases of horospherical varieties (which are toric bundles over flag varieties, at least after blow-ups); wonderful compactifications of semi-simple groups are special cases of wonderful varieties. This approach has also been applied to some non-spherical varieties, namely equivariant compactifications of vector groups [CLT02] and Cayley's singular ruled cubic surface [BBS16b].

The universal torsor method for Manin's conjecture was initiated by Salberger [Sal98], who gave a new proof of Manin's conjecture for split toric varieties over \mathbb{Q} , which are spherical. Moreover, estimating rational points on a projective variety $X \subseteq \mathbb{P}^n$ by counting integral points on its affine cone in \mathbb{A}^{n+1} , e.g., by the circle method [Bir62], can be interpreted as an instance of the universal torsor method. However, all other applications of the universal torsor method seem to concern non-spherical varieties. In dimension 2, there are many examples of smooth and singular del Pezzo surfaces with a crepant resolution; see [Bre02, BBP12, BBD07, BB13], for example. In higher dimension, only three cases are known so far: Segre's singular cubic threefold [Bre07], a singular cubic fourfold [BBS14] and a singular biprojective cubic threefold [BBS16a]; in all three cases, the singularities have a crepant resolution. Hence all results proved by the universal torsor method are explained by Peyre's relatively classical version of Manin's conjecture [Pey03, 5.1].

The goal of our project is to start the investigation of Manin's conjecture for spherical varieties by the universal torsor method. For this method, an explicit description of the universal torsors is needed; this can be obtained from the Cox rings of the underlying varieties (for details, see [DP14], for example). Cox rings of spherical varieties were determined by Brion [Bri07]. Also note that our results below are the first applications of the universal torsor method to varieties without a crepant resolution, where the more general conjectures of Batyrev and Tschinkel [BT98b] are relevant.

1.2. A singular weighted cubic threefold and (2×2) -determinants that are cubes. One of the simplest spherical varieties that is neither horospherical nor wonderful has the following nice and easy description: It is the singular weighted cubic threefold

$$X_2 := \mathbb{V}(ad - bc - z^3) \subseteq Y_2 := \mathbb{P}_{\mathbb{Q}}(1, 2, 1, 2, 1)$$

in the weighted projective space Y_2 with weighted homogeneous coordinates $(a : b : c : d : z)$. It is closely related to the following Diophantine problem: How often is the determinant of a (2×2) -matrix a cube? The question of representing a fixed number as a determinant over \mathbb{Z} is considered in [DRS93].

The action of the reductive group $\mathrm{SL}_2 \times \mathbb{G}_m$ defined by

$$\left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, t \right) \cdot \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) := \left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, z \right).$$

turns X_2 into a spherical variety. Its geometry can be analyzed by spherical techniques; we will do this in Section 3. For this introduction, we emphasize a weighted-projective point of view; see [Dol82].

Since $-K_{X_2} = \mathcal{O}_{X_2}(4)$, we obtain an anticanonical height

$$H : X_2(\mathbb{Q}) \rightarrow \mathbb{R}_{>0}$$

defined by

$$H(a : b : c : d : z) := \frac{\max\{|a^4|, |b^2|, |c^4|, |d^2|, |z^4|\}}{\gcd(a^4, b^2, c^4, d^2, z^4)}$$

for $a, b, c, d, z \in \mathbb{Z}$; note that in weighted projective space, we may not assume that the coordinates are coprime. See also Section 4.

Blowing up its singular locus $\mathbb{V}(a, c, z) \cong \mathbb{P}_{\mathbb{Q}}^1$ gives a crepant resolution $\pi : \tilde{X}_2 \rightarrow X_2$ (in particular, X_2 has at worst canonical singularities), with $\mathrm{Pic}(\tilde{X}_2)$ free of rank 2. This means that we are in the situation of Peyre's relatively classical version [Pey03, 5.1] of Manin's conjecture. Our first main result (see Theorem 8.6 for its proof) is compatible with this prediction (see Section 5):

Theorem 1.1. *We have*

$$N_{X_2, H}(B) = \mathfrak{c} B \log B + O(B),$$

where

$$\mathfrak{c} = \frac{1}{8} \cdot \frac{1}{\zeta(2)\zeta(3)} \cdot \left(2 \iiint \int_{|a|, |c|, |z|, |(ad-z^3)/c|, |d| \leq 1} \frac{1}{|c|} da dc dd dz \right)$$

is Peyre's constant.

1.3. A family of spherical threefolds. Our weighted cubic threefold $X_2 \subseteq \mathbb{P}_{\mathbb{Q}}(1, 2, 1, 2, 1)$ can be generalized as follows. For any positive integer n , consider the weighted hypersurface

$$X_n := \mathbb{V}(ad - bc - z^{n+1}) \subseteq Y_n := \mathbb{P}_{\mathbb{Q}}(1, n, 1, n, 1)$$

of degree $n+1$ in the weighted projective space Y_n with weighted homogeneous coordinates $(a : b : c : d : z)$. With an action of $\mathrm{SL}_2 \times \mathbb{G}_m$ that has the same description as above for X_2 , each X_n is a spherical threefold that is neither horospherical nor wonderful.

Let $n \geq 3$. By choosing sections of the very ample $\frac{n}{n+2}$ -th power of the \mathbb{Q} -Cartier divisor $-K_{X_n} = \mathcal{O}_{X_n}(n+2)$, we obtain an anticanonical height

$$H : X_n(\mathbb{Q}) \rightarrow \mathbb{R}_{>0}$$

defined by

$$H(a : b : c : d : z) = \left(\frac{\max\{|a^n|, |b|, |c^n|, |d|, |z^n|\}}{\gcd(a^n, b, c^n, d, z^n)} \right)^{\frac{n+2}{n}}$$

for $a, b, c, d, z \in \mathbb{Z}$; see also Section 4.

Naïve heuristic considerations ignoring the denominator of the height function (analogous to the ones in [HB07, Heuristic principle] and [BT98b, §5.1]) lead to

the expectation that $N_{X_n, H}(B)$ might grow linearly. However, in our second main result, we show (see Theorem 8.4 for its proof):

Theorem 1.2. *Let $n \geq 3$. We have*

$$N_{X_{n, \text{reg}}, H}(B) = \left(\sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbb{V}(a, c)} \mathfrak{c}_x \right) B^{\frac{2n}{n+2}} + O(B),$$

where $X_{n, \text{reg}}$ denotes the smooth locus of X_n . The value of the leading constant is

$$\mathfrak{c}_x = \frac{1}{2} \cdot \frac{1}{\zeta(2)} \cdot \omega_{\infty, x},$$

where (assuming that a, c, z are coprime integral coordinates for x)

$$\omega_{\infty, x} = \begin{cases} \iint \frac{1}{|a|} db dw & \text{for } a \neq 0, \\ \iint \frac{1}{|c|} dd dw & \text{for } c \neq 0. \end{cases}$$

We will see that $X_{n, \text{reg}}$ is covered by rational curves, each of which contains $\sim \mathfrak{c}_x B^{2n/(n+2)}$ rational points of height at most B . Therefore, we cannot obtain linear growth by removing a closed or thin subset.

Instead, we discuss in the next part of this introduction how our result is explained by the predictions of Batyrev–Tschinkel [BT98b]; see Section 6 for more details. Note that the singular locus $X_{n, \text{sing}}$ is a weakly accumulating subvariety, with $N_{X_{n, \text{sing}}, H}(B) \sim \frac{2}{\zeta(2)} B^{2n/(n+2)}$ (see Remark 4.5); we exclude it in Theorem 1.2 to obtain a result that is compatible with [BT98b].

1.4. The predictions of Batyrev–Tschinkel. Let X be a Fano variety over \mathbb{Q} with at worst log terminal singularities and a Zariski dense set of rational points. Let $H: X(\mathbb{Q}) \rightarrow \mathbb{R}_{\geq 0}$ be an anticanonical height function. Let $\pi: \tilde{X} \rightarrow X$ be a desingularization and $L := \pi^*(-K_X)$. By [BM90, Conjecture C'] and [Pey03, 3.6], we expect

$$N_{U, H}(B) := \#\{x \in U(\mathbb{Q}) : H(x) \leq B\} \sim \mathfrak{c} B^{\mathfrak{a}} (\log B)^{\mathfrak{b}-1}$$

as $B \rightarrow \infty$, where U is the complement of the closed (or thin) subset consisting of the accumulating subvarieties, $\mathfrak{a} := \inf\{t \in \mathbb{R} : t \cdot L + K_{\tilde{X}} \text{ is effective}\}$ and \mathfrak{b} is the codimension of the minimal face of the effective cone of \tilde{X} containing $\mathfrak{a} \cdot L + K_{\tilde{X}}$. Note that the effective cone of a Fano variety with log terminal singularities is simplicial by [BCHM10, Corollary 1.3.2]. If X has at worst canonical singularities, then $L + K_{\tilde{X}}$ is effective, hence $\mathfrak{a} \leq 1$. On the other hand, for varieties with worse singularities, we may have $\mathfrak{a} > 1$, in which case more than linear growth is expected.

A prediction for the leading constant \mathfrak{c} is given in [BT98b]. Here, one considers the \mathcal{L} -primitive fibration (see [BT98b, Definition 2.4.2])

$$\phi: X \dashrightarrow P := \text{Proj} \left(\bigoplus_{\nu \geq 0} \Gamma \left(\tilde{X}, (\mathfrak{a} \cdot L + K_{\tilde{X}})^{\otimes \nu} \right) \right),$$

and, for some restriction to open subsets $\phi: U \rightarrow V$, the constant \mathfrak{c} is given by

$$\sum_{x \in V} \mathfrak{c}_x,$$

where \mathfrak{c}_x is the expected constant in the asymptotic formula for the fiber $\phi^{-1}(x)$. The sum should be taken over the fibers that contain a positive proportion of the rational points (these are called \mathcal{L} -targets, see [BT98b, Definition 3.2.4]). If

the divisor $\mathfrak{a} \cdot L + K_{\widetilde{X}}$ is *rigid* ([BT98b, Definition 2.3.1], e. g., if X has a crepant resolution), then the variety P is a point.

Batyrev and Tschinkel make the following prediction in [BT98b, Conjecture 3.5.1]:

Conjecture 1.3. *Let \overline{H} be a height on P relative to the line bundle $\mathcal{O}_P(-1) \otimes \omega_P$. Then there exist positive constants c_1, c_2 and an open subset $V \subseteq P$ such that for every $x \in V$ we have*

$$c_1 \overline{H}(x) \leq \mathfrak{c}_x \leq c_2 \overline{H}(x).$$

We apply the conjectures of [BT98b] to our family X_n of spherical varieties; see Section 3 for their geometry. Blowing up the singular locus $\mathbb{V}(a, c, z) \cong \mathbb{P}_{\mathbb{Q}}^1$ gives a desingularization $\pi: \widetilde{X}_n \rightarrow X_n$, and we will see that we have

$$\mathfrak{a} = \frac{2n}{n+2} \quad \text{and} \quad \mathfrak{b} = \begin{cases} 2, & \text{for } n = 2, \\ 1, & \text{for } n \geq 3. \end{cases}$$

For $n \geq 3$, the singularities of X_n are not canonical, but log terminal. The divisor $\mathfrak{a} \cdot \pi^*(-K_{X_n}) + K_{\widetilde{X}_n}$ is not rigid, and the \mathcal{L} -primitive fibration turns out to be a map $\phi: X_n \dashrightarrow P_n \cong \mathbb{P}_{\mathbb{Q}}^2$ with $\mathcal{O}_{P_n}(1) \cong \mathcal{O}_{\mathbb{P}_{\mathbb{Q}}^2}(n-2)$ such that the constants \mathfrak{c}_x appearing in Theorem 1.2 are Peyre's constant for the fibers $\phi^{-1}(x)$.

In the proofs of Theorems 1.1 and 1.2, we work with universal torsors over a further blow-up $\widehat{X}_n \rightarrow \widetilde{X}_n \rightarrow X_n$ because this leads to more convenient coprimality conditions in the associated counting problem (see Remark 4.4). This seems surprising to us because proofs of cases of Manin's conjecture for singular del Pezzo surfaces usually use universal torsors of their minimal desingularizations.

It turns out that Conjecture 1.3 of Batyrev–Tschinkel is true for X_n (see Theorem 6.3):

Theorem 1.4. *Let $\overline{H}: \mathbb{P}^2(\mathbb{Q}) \rightarrow \mathbb{R}_{>0}$ be a height relative to*

$$\mathcal{O}_{\mathbb{P}_{\mathbb{Q}}^2}(-n-1) \cong \mathcal{O}_{P_n}(-1) \otimes \omega_{P_n}.$$

There exist positive constants c_1, c_2 such that for every $x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbb{V}(a, c)$ we have

$$c_1 \overline{H}(x) \leq \mathfrak{c}_x \leq c_2 \overline{H}(x).$$

This implies that the sum over the constants \mathfrak{c}_x in Theorem 1.2 converges.

1.5. A second family of spherical threefolds. Since the varieties X_n considered above are equivariant compactifications of \mathbb{G}_a^3 , Manin's conjecture is already known for them by [CLT02] (for heights corresponding to smooth adelic metrics; note that we work with a height corresponding to an adelic metric that is not smooth). To illustrate that our approach can also be applied to spherical varieties without such a structure, we consider a family of varieties X'_n for $n \geq 2$ that do not belong to any of the classes of varieties for which Manin's conjecture is known.

A comparison of the geometric description, the shape of the main results and their proofs for the family X_n with the family X'_n will reveal many similarities, but also several additional complications for X'_n . In particular, we will see that Conjecture 1.3 fails for X'_n . Hence the family X_n can be regarded as a warm-up for the family X'_n .

Fix an integer $n \geq 2$. Consider the weighted projective space Y_{n-1} with Cox coordinates $(a : b : c : d : y)$ and the toric modification $Y'_n \rightarrow Y_{n-1}$ obtained by first blowing up the singular locus of Y_{n-1} , then blowing up the two torus invariant curves in the resulting exceptional divisor, and finally contracting the exceptional divisor from the first step. With Cox coordinates $(a : b : c : d : y : z : t)$, where z corresponds to the torus invariant curve in Y'_{n-1} contained in $\mathbb{V}(y)$ and t to the other one, we consider the hypersurface

$$X'_n := \mathbb{V}(ad - bc - y^n z^{n+1}) \subseteq Y'_n.$$

Equipped with a suitable action of the reductive group $\mathrm{SL}_2 \times \mathbb{G}_m$, it is a singular spherical threefold that is neither horospherical nor wonderful; moreover it is not isomorphic to an equivariant compactification of \mathbb{G}_a^3 since its effective cone can be shown not to be simplicial.

In Section 3, we will construct a desingularization $\pi: \tilde{X}'_n \rightarrow X'_n$, and we will see that we have

$$\mathfrak{a} = \frac{2n+2}{n+3} \quad \text{and} \quad \mathfrak{b} = 1.$$

In Section 4, we will construct an anticanonical height

$$H': X'_n \rightarrow \mathbb{R}_{>0}$$

by choosing sections of a very ample power of the \mathbb{Q} -Cartier divisor $\pi^*(-K_{X'_n})$ on \tilde{X}'_n .

The singularities of X'_n are log terminal, and the divisor $\mathfrak{a} \cdot \pi^*(-K_{X'_n}) + K_{\tilde{X}'_n}$ is not rigid. We will find the \mathcal{L} -primitive fibration $\phi': X'_n \dashrightarrow P'_n \cong \mathbb{P}^2_{\mathbb{Q}}$, where we denote the homogeneous coordinates of $\mathbb{P}^2_{\mathbb{Q}}$ by $(\hat{a} : \hat{c} : \hat{y})$. Again, our main result (see Theorem 9.1) is compatible with the predictions of [BT98b] (see Section 7).

Theorem 1.5. *Let $n \geq 2$ and $U' := X'_n \setminus \mathbb{V}(yzt)$. For every $\epsilon > 0$, we have*

$$N_{U', H'}(B) = \left(\sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus (\mathbb{V}(\hat{a}, \hat{c}) \cup \mathbb{V}(\hat{y}))} \mathfrak{c}_x \right) B^{\frac{2n+2}{n+3}} + O_{\epsilon}(B^{1+\epsilon}),$$

where each summand \mathfrak{c}_x in the leading constant is Peyre's constant for the rational fiber $\phi'^{-1}(x)$. Its value is

$$\mathfrak{c}_x = \frac{1}{2} \cdot \left(\prod_{p \text{ prime}} \left(1 - \frac{1}{p} \right) \omega_{p,x} \right) \cdot \omega_{\infty, x},$$

with (assuming that $\hat{a}, \hat{c}, \hat{y}$ are coprime integral coordinates for x and $\mathfrak{c} := \frac{-n+1}{n+3}$)

$$\omega_{p,x} = \left(\left(1 - \frac{1}{p} \right) \cdot \frac{1 - (p^{\epsilon})^{\nu_p(\hat{y})+1}}{1 - p^{\epsilon}} + \frac{1}{p} + \frac{(p^{\epsilon})^{\nu_p(\hat{y})}}{p} \right) \cdot (p^{\epsilon+1})^{\min(\nu_p(\hat{a}), \nu_p(\hat{c}))},$$

and

$$\omega_{\infty, x} = \begin{cases} \iint_{\max |\mathcal{M}'_n(\hat{a}, b, \hat{c}, (b\hat{c} + \hat{y}^n)/\hat{a}), \hat{y}, 1, 1, w)| \leq 1} \frac{1}{|\hat{a}|} db dw & \text{for } \hat{a} \neq 0, \\ \iint_{\max |\mathcal{M}'_n(\hat{a}, (\hat{a}d - \hat{y}^n)/\hat{c}, \hat{c}, d, \hat{y}, 1, 1, w)| \leq 1} \frac{1}{|\hat{c}|} db dw & \text{for } \hat{c} \neq 0, \end{cases}$$

where $\mathcal{M}'_n(\dots)$ denotes the set of 13 monomials from Remark 4.8.

In particular, the expressions for the p -adic densities $\omega_{p,x}$ are apparently much more complicated than in previous applications of the universal torsor method for Manin's conjecture. Also note that $\omega_{p,x}$ depends on the base point x , while the p -adic densities in Theorem 1.2 are independent of x .

Finally, Conjecture 1.3 of Batyrev–Tschinkel is not true for X'_n (see Theorem 7.3):

Theorem 1.6. *Let $\overline{H}: \mathbb{P}^2(\mathbb{Q}) \rightarrow \mathbb{R}_{>0}$ be a height relative to an arbitrary line bundle. There does not exist an open subset $V \subseteq \mathbb{P}^2(\mathbb{Q})$ with positive constants c_1, c_2 such that for every $x \in V$ we have*

$$c_1 \overline{H}(x) \leq \mathfrak{c}_x \leq c_2 \overline{H}(x).$$

Nevertheless, we can show that the sum over the constants \mathfrak{c}_x in Theorem 1.5 converges (see Proposition 9.3).

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2. SPHERICAL VARIETIES WITH ACTING GROUP OF SEMISIMPLE RANK 1

In this section, we give an overview of the classification of spherical varieties and their Cox rings, focusing on the simplest case of semisimple rank 1. Where our families X_n and X'_n appear in this classification will be explained in the following section.

Let G be a connected reductive group over $\overline{\mathbb{Q}}$, and let $B \subseteq G$ be a Borel subgroup containing a maximal torus T . A normal G -variety X over $\overline{\mathbb{Q}}$ is called *spherical* if it contains a dense B -orbit. By passing to a finite cover, it is always possible to replace G with a group of the form $G^{ss} \times C$ where G^{ss} is semisimple simply-connected and C is a torus. The *semisimple rank* of G is defined to be the rank of G^{ss} . Spherical varieties with acting group of semisimple rank 0, i.e., the case where G^{ss} is trivial, are toric varieties. In this paper, we will be concerned with examples of spherical varieties with acting group of semisimple rank 1, i.e., we have $G^{ss} = \mathrm{SL}_2$.

Spherical varieties admit a combinatorial description generalizing the combinatorial description of toric varieties. The (unique) open G -orbit G/H in X is called a *spherical homogeneous space* and replaces the dense torus (in contrast to the toric case, it is not possible to assume that H is trivial). Hence, the first step of the combinatorial description of spherical varieties is, given a connected reductive group G , to describe all spherical homogeneous spaces G/H (i.e., all *spherical subgroups* $H \subseteq G$). This is achieved by a program initiated by Luna (see [Lun01]), which has been completed recently (see [BP15, CF14, Los09]). Then, given a spherical homogeneous space G/H , one describes all G -equivariant open embeddings $G/H \hookrightarrow X$ into a normal irreducible G -variety X . This is achieved by the Luna–Vust theory (see [LV83, Kno91]) in terms of so-called *colored fans*, which generalize the usual fans of toric varieties. We also refer to [BL11, Per14, Tim11] as general references.

For a spherical homogeneous space, the *weight lattice* $\mathcal{M} \subseteq \mathfrak{X}(B)$, which is the set of weights of B -semi-invariants (i.e., B -eigenvectors) occurring in $\overline{\mathbb{Q}}(G/H)$, replaces the character lattice in the toric case. There are two further combinatorial objects to consider (which are both trivial in the toric case). The first is the *set of colors* \mathcal{D} , which is the set of B -invariant prime divisors in G/H . The second is the *valuation cone* $\mathcal{V} \subseteq \mathcal{N}_{\mathbb{Q}} := \mathrm{Hom}(\mathcal{M}, \mathbb{Q})$, which can be identified with the \mathbb{Q} -valued G -invariant discrete valuations on $\overline{\mathbb{Q}}(G/H)$. Spherical varieties with $\mathcal{V} = \mathcal{N}_{\mathbb{Q}}$ are called *horospherical*. The set \mathcal{D} is equipped with the map $\rho: \mathcal{D} \rightarrow \mathcal{N} := \mathrm{Hom}(\mathcal{M}, \mathbb{Z})$ defined by $\langle \rho(D), \chi \rangle := \nu_D(f_\chi)$ where ν_D is the valuation on $\overline{\mathbb{Q}}(G/H)$ which is induced by the prime divisor D and $f_\chi \in \overline{\mathbb{Q}}(G/H)$ is a B -semi-invariant of weight $\chi \in \mathcal{M}$ (which is defined up to a constant factor because of the open B -orbit).

According to the Luna program, the spherical subgroup $H \subseteq G$ is (roughly speaking) determined by the combinatorial objects \mathcal{M} , \mathcal{D} , and \mathcal{V} , where an important step is to reduce this description to the case of so-called *spherically closed* spherical subgroups. The spherical closure \overline{H} of a spherical subgroup $H \subseteq G$ is the greatest group between H and $N_G(H)$ such that the natural map $G/H \rightarrow G/\overline{H}$ induces a bijection of B -invariant prime divisors. In the case where $G = \mathrm{SL}_2 \times C$ is of semisimple rank 1, we have the four cases

$$\overline{H} = T_{\mathrm{SL}_2} \times C, \quad \overline{H} = N_{\mathrm{SL}_2} \times C, \quad \overline{H} = B_{\mathrm{SL}_2} \times C, \quad \overline{H} = \mathrm{SL}_2 \times C,$$

where T_{SL_2} , N_{SL_2} , and B_{SL_2} denote a maximal torus, its normalizer, and a Borel subgroup of SL_2 , respectively. We ignore the last case, which reduces to G of semisimple rank 0. We abbreviate the other cases by T , N , and B . Let $\alpha \in \mathfrak{X}(T) = \mathfrak{X}(B)$ be the unique simple root.

- *Case T.* The lattice \mathcal{M} contains $\gamma := \alpha$ as a primitive element. There are exactly two colors $D', D'' \in \mathcal{D}$ with $\rho(D') + \rho(D'') = \alpha^\vee|_{\mathcal{M}}$ where $\alpha^\vee: \mathfrak{X}(B) \rightarrow \mathbb{Z}$ denotes the coroot to $\alpha \in \mathfrak{X}(B)$. The valuation cone is the half-space $\mathcal{V} = -\text{cone}(\gamma)^\vee \subseteq \mathcal{N}_{\mathbb{Q}}$.
- *Case N.* The lattice \mathcal{M} contains $\gamma := 2\alpha$ as a primitive element. There is exactly one color $D' \in \mathcal{D}$ with $\rho(D') = \frac{1}{2}\alpha^\vee|_{\mathcal{M}}$. The valuation cone is the half-space $\mathcal{V} = -\text{cone}(\gamma)^\vee \subseteq \mathcal{N}_{\mathbb{Q}}$.
- *Case B.* There is no condition on the lattice \mathcal{M} . There is exactly one color $D' \in \mathcal{D}$ with $\rho(D') = \alpha^\vee|_{\mathcal{M}}$. The valuation cone is the whole space $\mathcal{V} = \mathcal{N}_{\mathbb{Q}}$.

We now briefly describe the second step of the combinatorial description of spherical varieties, namely the description of spherical embeddings $G/H \hookrightarrow X$ of a fixed spherical homogeneous space G/H by colored fans. A colored fan is a set consisting of *colored cones*, which are pairs $(\mathcal{C}, \mathcal{F})$ where \mathcal{C} is a cone in $\mathcal{N}_{\mathbb{Q}}$ and \mathcal{F} is a subset of \mathcal{D} . Moreover, certain properties and compatibility conditions have to be satisfied. Similarly to the case of toric varieties, the colored cones are in bijection with the G -orbits in X . The colored cones corresponding to G -orbits of codimension 1 are easier to describe. They have the form (ρ, \emptyset) where ρ is a ray in \mathcal{V} , which means that we have $\rho = \text{cone}(u)$ for a uniquely determined primitive element $u \in \mathcal{V} \cap \mathcal{N}$.

Let $u_1, \dots, u_n \in \mathcal{V} \cap \mathcal{N}$ be the primitive elements corresponding to the G -invariant prime divisors D_1, \dots, D_n in X . According to [Bri07, Proposition 4.1.1], the divisor class group $\text{Cl}(X)$ is generated by divisor classes $[D_1], \dots, [D_n]$ and the divisor classes of the colors \mathcal{D} , and the relations can be computed from the relative position of the $u_1, \dots, u_n \in \mathcal{N}$ similarly to the toric case.

In the case where G is of semisimple rank 1, the Cox ring of X can be obtained explicitly using [Bri07, Theorem 4.3.2] or [Gag14, Theorem 3.6]:

Proposition 2.1. *In the case T, let $r_i := -\langle u_i, \gamma \rangle$. We obtain*

$$\mathcal{R}(X) = \overline{\mathbb{Q}}[a, b, c, d, z_1, \dots, z_n] / \langle ad - bc - z_1^{r_1} \dots z_n^{r_n} \rangle$$

with $\deg(a) = \deg(c) = [D']$, $\deg(b) = \deg(d) = [D'']$, and $\deg(z_i) = [D_i]$.

In the case N, let $r_i := -\langle u_i, \gamma \rangle$. We obtain

$$\mathcal{R}(X) = \overline{\mathbb{Q}}[a, b, c, z_1, \dots, z_n] / \langle ac - b^2 - z_1^{r_1} \dots z_n^{r_n} \rangle$$

with $\deg(a) = \deg(b) = \deg(c) = [D']$ and $\deg(z_i) = [D_i]$.

In the case B, we obtain

$$\mathcal{R}(X) = \overline{\mathbb{Q}}[a, b, z_1, \dots, z_n]$$

with $\deg(a) = \deg(b) = [D']$ and $\deg(z_i) = [D_i]$.

In the case B, note that the Cox ring is a polynomial ring, hence that the variety X is isomorphic to a toric variety. In particular, every horospherical variety with acting group of semisimple rank 1 is isomorphic to a toric variety.

Remark 2.2. Spherical varieties with acting group of semisimple rank 1 admit a torus action of complexity one (by considering a maximal torus in G). The class of spherical varieties with acting group of semisimple rank 1 coincides with the class of quasihomogeneous rational varieties with torus action of complexity one. For details, we refer to [AHHL14], where the Cox rings of rational varieties with torus action of complexity one have been described.

3. TWO FAMILIES OF SPHERICAL HYPERSURFACES IN TORIC VARIETIES

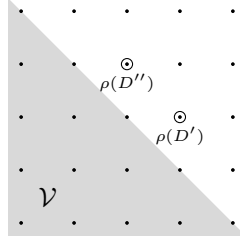
For $G := \text{SL}_2$ the spherical G -varieties are at most 2-dimensional, where each complete one is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, \mathbb{P}^2 , or the blow-up of \mathbb{P}^2 in one point. The

next possible step is to consider $G := \mathrm{SL}_2 \times \mathbb{G}_m$. Of the two (non-horospherical) types T and N (as introduced in Section 2), we will consider the type T . The simplest spherical subgroups of type T are $H := T_{\mathrm{SL}_2} \times \mathbb{G}_m$, which reduces to the case $G = \mathrm{SL}_2$, and $H := T_{\mathrm{SL}_2} \times \{e\}$, where every embedding $G/H \hookrightarrow X$ has $\mathrm{rank} \mathrm{Cl}(X) \geq 2$.

In order to find examples with $\mathrm{rank} \mathrm{Cl}(X) = 1$ which do not reduce to the case $G = \mathrm{SL}_2$, we choose a slightly more complicated spherical subgroup $H \subseteq G$. Let $\varepsilon : \mathbb{G}_m \rightarrow \mathbb{G}_m$ be a primitive character and consider the spherical subgroup

$$H := \{(\lambda, \varepsilon(\lambda)) : \lambda \in T_{\mathrm{SL}_2}\} \subseteq G.$$

Using the notation from Section 2, the lattice \mathcal{M} has basis $(\frac{1}{2}\alpha + \varepsilon, \frac{1}{2}\alpha - \varepsilon)$, the corresponding dual basis of the lattice \mathcal{N} is $(\rho(D'), \rho(D''))$, and for the valuation cone we have $\mathcal{V} = \{v \in \mathcal{N}_{\mathbb{Q}} : \langle v, \alpha \rangle \leq 0\}$. The situation inside the vector space $\mathcal{N}_{\mathbb{Q}}$ is illustrated in the following picture.



For every $n \geq 2$, we consider the spherical embedding $G/H \hookrightarrow X_n$ with exactly one G -invariant prime divisor corresponding to the primitive element

$$u_z := -\rho(D') - n\rho(D'') \in \mathcal{V} \cap \mathcal{N}.$$

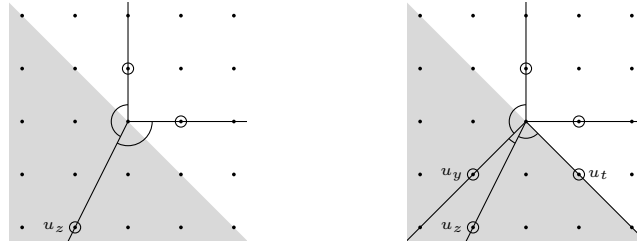
It can be shown that X_n is isomorphic to an equivariant compactification of \mathbb{G}_a^3 .

We therefore consider the spherical embedding $G/H \hookrightarrow X'_n$ with two additional G -invariant prime divisors corresponding to the primitive elements

$$u_y := -\rho(D') - (n-1)\rho(D'') \in \mathcal{V} \cap \mathcal{N},$$

$$u_t := \rho(D') - \rho(D'') \in \mathcal{V} \cap \mathcal{N}.$$

It can be shown that the effective cone of X'_n is not simplicial, hence X'_n is not isomorphic to an equivariant compactification of \mathbb{G}_a^3 . The colored fans of X_2 and X'_2 are illustrated in the following pictures.



Using [DP14], we consider X_n and X'_n as varieties over \mathbb{Q} . According to Proposition 2.1, we have

$$\mathcal{R}(X_n) = \mathbb{Q}[a, b, c, d, z] / \langle ad - bc - z^{n+1} \rangle$$

with $\mathrm{Cl}(X_n) \cong \mathbb{Z}$ where $\deg(a) = \deg(c) = \deg(z) = 1$ and $\deg(b) = \deg(d) = n$. Moreover, the graded ring $\mathbb{Q}[a, b, c, d, z]$, where we ignore the relation, is identified as the Cox ring of the weighted projective space $Y_n := \mathbb{P}_{\mathbb{Q}}(1, n, 1, n, 1)$. It follows that X_n is a hypersurface in Y_n defined by $ab - cd - z^{n+1} = 0$.

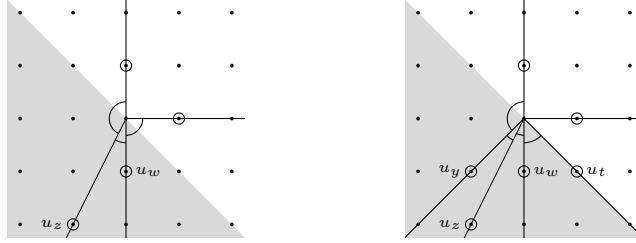
Similarly, we have

$$\mathcal{R}(X'_n) = \mathbb{Q}[a, b, c, d, y, z, t] / \langle ad - bc - y^n z^{n+1} \rangle$$

with $\text{Cl}(X'_n) \cong \mathbb{Z}^3$ where $\deg(a) = \deg(c) = (1, 1, -1)$, $\deg(b) = \deg(d) = (n, n - 1, 1)$, $\deg(z) = (1, 0, 0)$, $\deg(y) = (0, 1, 0)$, and $\deg(t) = (0, 0, 1)$. Again, the variety X'_n is a hypersurface in a toric variety Y'_n with graded Cox ring $\mathbb{Q}[a, b, c, d, y, z, t]$.

According to [Bri97, 4.1 and 4.2] or [ADHL15, Proposition 3.3.3.2], we have the anticanonical divisor classes $-K_{X_n} = n + 2$ and $-K_{X'_n} = (n + 2, n + 1, 1)$. Moreover, according to [GH15, Theorem 1.9] or [ADHL15, 3.3.2.9], the varieties X_n and X'_n are Fano for every $n \geq 2$, and the variety X_2 is Gorenstein.

The singular loci are $X_{n,\text{sing}} = X \cap \mathbb{V}(a, c, z)$ and $X'_{n,\text{sing}} = X' \cap \mathbb{V}(z, t)$. We construct desingularizations $\pi: \tilde{X}_n \rightarrow X_n$ and $\pi': \tilde{X}'_n \rightarrow X'_n$ by subdividing their colored fans. We add a G -invariant prime divisor corresponding to the primitive element $u_w := -\rho(D'') \in \mathcal{V} \cap \mathcal{N}$. The resulting colored fans of the spherical varieties \tilde{X}_2 and \tilde{X}'_2 are illustrated in the following pictures.



According to Proposition 2.1, we have

$$\mathcal{R}(\tilde{X}_n) = \mathbb{Q}[a, b, c, d, z, w] / \langle ad - bc - z^{n+1}w \rangle$$

with $\text{Pic}(\tilde{X}_n) \cong \text{Cl}(\tilde{X}_n) \cong \mathbb{Z}^2$ where $\deg(a) = \deg(c) = \deg(z) = (1, 0)$, $\deg(b) = \deg(d) = (n, 1)$, and $\deg(w) = (0, 1)$. Moreover, we have

$$\mathcal{R}(\tilde{X}'_n) = \mathbb{Q}[a, b, c, d, y, z, t, w] / \langle ad - bc - y^n z^{n+1}w \rangle$$

with $\text{Pic}(\tilde{X}'_n) \cong \text{Cl}(\tilde{X}'_n) \cong \mathbb{Z}^4$ where $\deg(a) = \deg(c) = (1, 1, -1, 0)$, $\deg(b) = \deg(d) = (n, n - 1, 1, 1)$, $\deg(z) = (1, 0, 0, 0)$, $\deg(y) = (0, 1, 0, 0)$, $\deg(t) = (0, 0, 1, 0)$, and $\deg(w) = (0, 0, 0, 1)$.

In order to obtain explicit descriptions of \tilde{X}_n and \tilde{X}'_n , we use [ADHL15, Theorem 2.2.2.2, Proposition 3.3.2.9, and Construction 3.2.1.3] and [DP14], according to which the quasi-affine varieties

$$\mathcal{T}_n := \text{Spec}(\mathcal{R}(\tilde{X}_n)) \setminus (\mathbb{V}(a, c, z) \cup \mathbb{V}(b, d, w)),$$

$$\mathcal{T}'_n := \text{Spec}(\mathcal{R}(\tilde{X}'_n)) \setminus (\mathbb{V}(a, c) \cup \mathbb{V}(b, d, z) \cup \mathbb{V}(b, d, w) \cup \mathbb{V}(y, w) \cup \mathbb{V}(y, t) \cup \mathbb{V}(z, t))$$

are universal torsors $\mathcal{T}_n \rightarrow \tilde{X}_n$ and $\mathcal{T}'_n \rightarrow \tilde{X}'_n$ with respect to the natural actions of the tori

$$\text{Spec}(\mathbb{Q}[\text{Pic}(\tilde{X}_n)]) \cong \mathbb{G}_m^2 \text{ and } \text{Spec}(\mathbb{Q}[\text{Pic}(\tilde{X}'_n)]) \cong \mathbb{G}_m^4$$

respectively.

According to [Bri97, 4.1 and 4.2] or [ADHL15, Proposition 3.3.3.2], we have

$$\begin{aligned} -K_{\tilde{X}_n} &= (n + 2, 2), & \pi^*(-K_{X_n}) &= (n + 2, \frac{n+2}{n}), \\ -K_{\tilde{X}'_n} &= (n + 2, n + 1, 1, 2), & \pi'^*(-K_{X'_n}) &= (n + 2, n + 1, 1, \frac{n+3}{n+1}). \end{aligned}$$

In particular, the resolution $\pi: \tilde{X}_n \rightarrow X_n$ is crepant and X_n has at worst canonical singularities if and only if $n = 2$ (see, for instance, [AB04]).

4. PARAMETERIZATION OF RATIONAL POINTS VIA UNIVERSAL TORSORS

Using the universal torsors \mathcal{T}_n and \mathcal{T}'_n from Section 3, we parameterize the rational points on X_n and X'_n , respectively.

Consider the line bundles

$$L := \frac{n}{n+2} \cdot \pi^*(-K_{X_n}) = (n, 1),$$

$$L' := (n+1) \cdot \pi^*(-K_{X'_n}) = (n^2 + 3n + 2, n^2 + 2n + 1, n+1, n+3).$$

We define

$$\mathcal{M}_n(a, b, c, d, z, w) := \{\text{monomials in } \mathcal{R}(\tilde{Y}_n) \text{ of degree } L \text{ restricted to } \tilde{X}_n\},$$

$$\mathcal{M}'_n(a, b, c, d, y, z, t, w) := \{\text{monomials in } \mathcal{R}(\tilde{Y}'_n) \text{ of degree } L' \text{ restricted to } \tilde{X}'_n\}.$$

Then we have

$$H(\pi(a : b : c : d : z : w)) := \left(\frac{\max |\mathcal{M}_n(a, b, c, d, z, w)|}{\gcd \mathcal{M}_n(a, b, c, d, z, w)} \right)^{(n+2)/n},$$

$$H'(\pi(a : b : c : d : y : z : t : w)) := \left(\frac{\max |\mathcal{M}'_n(a, b, c, d, y, z, t, w)|}{\gcd \mathcal{M}'_n(a, b, c, d, y, z, t, w)} \right)^{1/(n+1)}$$

for anticanonical heights H and X_n and H' on X'_n .

We are now going to state the counting problem for X_n . We consider the open subset

$$U := \tilde{X}_n \setminus \mathbb{V}(w) = X_n \setminus \mathbb{V}(a, c, z).$$

Proposition 4.1. *There is a natural 4-to-1 correspondence between*

$$\mathcal{U} := \left\{ (a, b, c, d, z, w) \in \mathbb{Z}^6 : \begin{array}{l} w \neq 0; \ ad - bc - z^{n+1}w = 0 \\ \gcd(a, c, z) = \gcd(b, d, w) = 1 \end{array} \right\}$$

and the set $U(\mathbb{Q})$. Moreover, for $(a, b, c, d, z, w) \in \mathcal{U}$, we have

$$H(\pi(a : b : c : d : z : w)) = \max |\mathcal{M}_n(a, b, c, d, z, w)|^{(n+2)/n}.$$

Proof. The toric variety \tilde{Y}_n comes from a regular fan. According to [Sal98, Section 8], we may construct a toric scheme \mathfrak{Y}_n over $\text{Spec}(\mathbb{Z})$, together with a map

$$\mathfrak{W}_n := \text{Spec}(\mathbb{Z}[a, b, c, d, z, w]) \setminus (\mathbb{V}(a, c, z) \cup \mathbb{V}(b, d, w)) \rightarrow \mathfrak{Y}_n,$$

which is a model for the universal torsor $\mathcal{W}_n \rightarrow \tilde{Y}_n$, obtain a 4-to-1 quotient

$$\mathfrak{W}_n(\mathbb{Z}) \rightarrow \mathfrak{Y}_n(\mathbb{Z}) = \tilde{Y}_n(\mathbb{Q})$$

for the $\mathbb{G}_m^2(\mathbb{Z}) \cong \{\pm 1\}^2$ -action as well as the claim on the height function. As we have

$$\mathfrak{W}_n(\mathbb{Z}) = \left\{ (a, b, c, d, z, w) \in \mathbb{Z}^6 : \gcd(a, c, z) = \gcd(b, d, w) = 1 \right\},$$

the result follows after restricting to the equation $ad - bc - z^{n+1}w = 0$.

Alternatively, the claims can easily be verified by elementary manipulations of the defining equation. \square

Corollary 4.2. *We have that $N_{U,H}(B^{(n+2)/n})$ is equal to*

$$\frac{1}{4} \# \left\{ (a, b, c, d, z, w) \in \mathbb{Z}^6 : \begin{array}{l} w \neq 0; \ ad - bc - z^{n+1}w = 0 \\ \gcd(a, c, z) = \gcd(b, d, w) = 1 \\ \max |\mathcal{M}_n(a, b, c, d, z, w)| \leq B \end{array} \right\}.$$

As the Diophantine equation $ad - bc = z^{n+1}w$ is easier to solve for d or b under the additional condition $\gcd(a, c) = 1$, we will use the following counting problem, which introduces an additional variable.

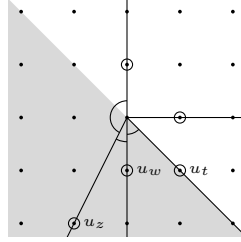
Corollary 4.3. *We have that $N_{U,H}(B^{(n+2)/n})$ is equal to*

$$\frac{1}{8} \# \left\{ (a, b, c, d, z, w, t) \in \mathbb{Z}^7 : \begin{array}{l} wt \neq 0; \quad ad - bc - z^{n+1}w = 0 \\ \gcd(a, c) = \gcd(b, d, w) = \gcd(z, t) = 1 \\ |a^n wt^{n+1}|, |c^n wt^{n+1}|, |z^n wt|, |b|, |d| \leq B \end{array} \right\}.$$

Remark 4.4. Corollary 4.3 can be interpreted as a version of Corollary 4.2, where instead of the desingularization $\tilde{X}_n \rightarrow X_n$, we use a further blow-up

$$\hat{X}_n \rightarrow \tilde{X}_n \rightarrow X_n.$$

The colored fan of \hat{X}_n is illustrated in the following picture (for $n = 2$).



According to Proposition 2.1, we have

$$\mathcal{R}(\tilde{X}_n) = \mathbb{Q}[a, b, c, d, z, w, t] / \langle ad - bc - z^{n+1}w \rangle$$

with $\text{Pic}(\tilde{X}_n) \cong \mathbb{Z}^3$ where $\deg(z) = (1, 0, 0)$, $\deg(w) = (0, 1, 0)$, $\deg(t) = (0, 0, 1)$, $\deg(a) = \deg(c) = (1, 0, -1)$, and $\deg(b) = \deg(d) = (n, 1, 1)$. Moreover, the quasi-affine variety

$$\mathcal{T}_n := \text{Spec}(\mathcal{R}(\tilde{X}_n)) \setminus (\mathbb{V}(a, c) \cup \mathbb{V}(b, d, w) \cup \mathbb{V}(z, t))$$

admits a torsor $\mathcal{T}_n \rightarrow \tilde{X}_n$ for the action of the torus $\text{Spec}(\mathbb{Q}[\text{Pic}(\tilde{X}_n)]) \cong \mathbb{G}_m^3$.

Remark 4.5. On the singular locus $X_{n,\text{sing}} = \mathbb{V}(a, c, z) \cong \mathbb{P}_{\mathbb{Q}}^1$ with coordinates $(b : d)$, the height H is the $\frac{n+2}{2n}$ -th power of the standard anticanonical height on $\mathbb{P}_{\mathbb{Q}}^1$. Therefore, we have

$$N_{X_{n,\text{sing}}, H}(B) = \frac{2}{\zeta(2)} B^{\frac{2n}{n+2}} + O(B^{\frac{n}{n+2}} \log B).$$

In particular,

$$N_{X_2, H}(B) = N_{U, H}(B) + O(B).$$

We are now going to state the counting problem for X'_n . We consider the open subset

$$U' := \tilde{X}'_n \setminus \mathbb{V}(yzwt) = X'_n \setminus \mathbb{V}(yzt).$$

Proposition 4.6. *There is a natural 16-to-1 correspondence between*

$$\mathcal{U}' := \left\{ (a, b, c, d, y, z, t, w) \in \mathbb{Z}^8 : \begin{array}{l} yztw \neq 0; \quad ad - bc - y^n z^{n+1}w = 0 \\ \gcd(a, c) = \gcd(z, t) = \gcd(y, t) = 1 \\ \gcd(b, d, z) = \gcd(b, d, w) = \gcd(y, w) = 1 \end{array} \right\}$$

and the set $U'(\mathbb{Q})$. Moreover, for $(a, b, c, d, y, z, t, w) \in \mathcal{U}'$, we have

$$H'(\pi(a : b : c : d : y : z : t : w)) = \max |\mathcal{M}'_n(a, b, c, d, y, z, t, w)|^{1/(n+1)}.$$

Proof. As Proposition 4.1. □

Corollary 4.7. *We have that $N_{U',H'}(B^{1/(n+1)})$ is equal to*

$$\frac{1}{16} \# \left\{ (a, b, c, d, y, z, t, w) \in \mathbb{Z}^8 : \begin{array}{l} yztw \neq 0; \quad ad - bc - y^n z^{n+1} w = 0 \\ \gcd(a, c) = \gcd(z, t) = \gcd(y, t) = 1 \\ \gcd(b, d, z) = \gcd(b, d, w) = \gcd(y, w) = 1 \\ \max |\mathcal{M}'_n(a, b, c, d, y, z, t, w)| \leq B \end{array} \right\}.$$

Remark 4.8. It is not difficult to see that we may assume that $\mathcal{M}'_n(a, b, c, d, y, z, t, w)$ only contains the 13 monomials

$$\begin{array}{llllll} \{b, d\}^{n+3} & \cdot \{a, c\}^2 & \cdot y^2, & & & \\ \{b, d\}^{n+1} & \cdot \{a, c\}^{2n+2} & & \cdot t^{2n+2} & \cdot w^2, & \\ \{b, d\}^{n+1} & & \cdot y^{2n+2} & \cdot z^{2n+2} & \cdot w^2, & \\ & \{a, c\}^{n^2+2n+1} & & \cdot z^{n+1} & \cdot t^{n^2+3n+2} & \cdot w^{n+3}, \\ & & y^{n^2+2n+1} & \cdot z^{n^2+3n+2} & \cdot t^{n+1} & \cdot w^{n+3}, \end{array}$$

where the notation $\{b, d\}$ resp. $\{a, c\}$ means b or d resp. a or c .

5. THE EXPECTED FORMULA FOR X_2

The aim of this section is to determine the expected asymptotic formula for $N_{U,H}(B)$ where

$$U := \tilde{X}_2 \setminus \mathbb{V}(w) = X_2 \setminus \mathbb{V}(a, c, z).$$

The resolution $\pi: \tilde{X}_2 \rightarrow X_2$ is crepant, hence the pullback of H is an anticanonical height on \tilde{X}_2 . According to [BM90, Conjecture C'] and [Pey03, 5.1], we have the predicted asymptotic formula

$$N_{U,H}(B) \sim \alpha \beta \tau B \log B.$$

with

$$\alpha = \text{rk Pic}(\tilde{X}_2) \cdot \text{vol} \left\{ t \in \text{Eff}(\tilde{X}_2)^\vee : (t, -K_{\tilde{X}_2}) \leq 1 \right\}$$

where the volume is normalized such that $\text{Pic}(\tilde{X}_2)^\vee$ has covolume 1 in $\text{Pic}(\tilde{X}_2)^\vee_{\mathbb{R}}$. Under the identification $\text{Pic}(\tilde{X}_2) \cong \mathbb{Z}^2$ from Section 3, we have

$$\alpha = 2 \cdot \text{vol} \left\{ (t_1, t_2) \in \mathbb{R}_{\geq 0}^2 : 4t_1 + 2t_2 \leq 1 \right\} = \frac{1}{8}.$$

The cohomological constant β is

$$\beta = \# H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}((\tilde{X}_2)_{\overline{\mathbb{Q}}})) = 1$$

since \tilde{X}_2 is split. Finally, we determine the Tamagawa number τ . Consider the chart

$$\mathbb{A}_{\mathbb{Q}}^3 \rightarrow \tilde{X}_2, \quad (a, d, z) \mapsto (a : ad - z^3 : 1 : d : z : 1).$$

It follows from [Pey03, 4.6] and [Pey95, 2.2.1] that we have

$$\tau = \omega_\infty \left(\prod_{p \text{ prime}} \lambda_p \omega_p \right)$$

with $\lambda_p = (1 - p^{-1})^2$ and

$$\omega_\nu := \iiint_{\mathbb{Q}_\nu^3} \frac{1}{\max |\mathcal{M}_2(a, ad - z^3, 1, d, z, 1)|_\nu^2} da dd dz,$$

for $\nu = p$ and $\nu = \infty$, where we have used the isomorphism

$$\omega_{\tilde{X}_2} \cong \mathcal{O}_{\tilde{X}_2}(-4, -2)$$

identifying section $da \wedge db \wedge dz$ from the chart with the section $1/c^4 w^2$ from the Cox ring. Note that ω_p and ω_∞ (but not the product τ) depend on the choice of such an isomorphism.

Lemma 5.1. *We have*

$$\prod_{p \text{ prime}} \lambda_p \omega_p = \frac{1}{\zeta(2)\zeta(3)}.$$

Proof. A direct calculation of the p -adic integral yields

$$\omega_p = \left(1 + \frac{1}{p}\right) \left(1 + \frac{1}{p} + \frac{1}{p^2}\right),$$

from which the result follows. Alternatively, we may compute ω_p using an integral model of \tilde{X}_2 . The toric variety \tilde{Y}_2 comes from a regular fan, which can be used to construct a toric scheme $\tilde{\mathfrak{Y}}_2$ over $\text{Spec}(\mathbb{Z})$, together with a map

$$\text{Spec}(\mathbb{Z}[a, b, c, d, z, w]) \setminus (\mathbb{V}(a, c, z) \cup \mathbb{V}(b, d, w)) \rightarrow \tilde{\mathfrak{Y}}_2,$$

which is a model for the universal torsor over

$$\tilde{Y}_2 = \tilde{\mathfrak{Y}}_2 \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q}).$$

For details, we refer to [Sal98, Section 8]. It can now be verified that the equation $ab - cd - z^{n+1}w = 0$ defines a closed subscheme $\tilde{\mathfrak{X}}_2 \hookrightarrow \tilde{\mathfrak{Y}}_2$, which is smooth and has integral fibers over $\text{Spec}(\mathbb{Z})$ such that

$$\tilde{X}_2 = \tilde{\mathfrak{X}}_2 \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q}).$$

We have the chart

$$\mathbb{A}_{\mathbb{Z}}^3 \rightarrow \tilde{\mathfrak{X}}_2, \quad (a, d, z) \mapsto (a : ad - z^3 : 1 : d : z : 1)$$

and see that the isomorphism of line bundles

$$\omega_{\tilde{\mathfrak{X}}_2} \cong \mathcal{O}_{\tilde{\mathfrak{X}}_2}(-4, -2)$$

identifying $da \wedge db \wedge dz$ and $1/c^4 w^2$ can be defined over $\text{Spec}(\mathbb{Z})$. Hence, according to [Pey16, Lemme 6.1], we have

$$\omega_p = \frac{\#\tilde{\mathfrak{X}}_2(\mathbb{F}_p)}{p^2} = \frac{(p+1)(p^2+p+1)}{p^2} = \left(1 + \frac{1}{p}\right) \left(1 + \frac{1}{p} + \frac{1}{p^2}\right). \quad \square$$

Finally, we compute the real density.

Lemma 5.2. *We have*

$$\omega_\infty = 2 \iiint \int_{|a|, |c|, |z|, |(ad-z^3)/c|, |d| \leq 1} \frac{1}{|c|} da dc dd dz.$$

Proof. We have

$$\omega_\infty = \iiint \frac{1}{\max\{|a^2|, |1|, |z^2|, |ad-z^3|, |d|\}^2} da dd dz.$$

We introduce an additional integration over c using the identity

$$\frac{1}{s} = \frac{1}{2} \int_{|c| \geq s} \frac{1}{|c^2|} dc$$

for $s \in \mathbb{R}_{>0}$ and obtain

$$\omega_\infty = \frac{1}{2} \iiint \int_{|a^2|, |1|, |z^2|, |ad-z^3|, |d| \leq |c|^{1/2}} \frac{1}{|c^2|} da dc dd dz.$$

Now the transformation $c \mapsto \frac{1}{c^4}$ (with $dc \mapsto \frac{4}{|c|^5} dc$) yields

$$\omega_\infty = 2 \iiint\limits_{|a^2 c^2|, |c^2|, |z^2 c^2|, |(ad-z^3)c^2|, |dc^2| \leq 1} |c^3| da dc dd dz,$$

and, finally, the transformation $(a, d, z) \mapsto (\frac{a}{c}, \frac{d}{c^2}, \frac{z}{c})$ yields

$$\omega_\infty = 2 \iiint\limits_{|a^2|, |c^2|, |z^2|, |(ad-z^3)/c|, |d| \leq 1} \frac{1}{|c|} da dc dd dz. \quad \square$$

6. THE EXPECTED FORMULA FOR X_n IN THE CASE $n \geq 3$

The aim of this section is to determine, for $n \geq 3$, the expected asymptotic formula for $N_{U,H}(B)$, where

$$U := \tilde{X}_n \setminus \mathbb{V}(w) = X_n \setminus \mathbb{V}(a, c, z)$$

and, moreover, to prove Theorem 1.4.

Recall from Section 4 that we consider

$$L := \frac{n}{n+2} \cdot \pi^*(-K_{X_n}) = (n, 1) \in \mathbb{Z}^2 \cong \text{Pic}(\tilde{X}_n),$$

and that the pullback of $H^{n/(n+2)}$ is a height relative to L . According to [BM90, Conjecture C'] (see also [Pey03, 3.6]), the predicted asymptotic formula is

$$N_{U,H}(B) \sim \mathfrak{c} B^{\mathfrak{a}} (\log B)^{\mathfrak{b}-1},$$

where

$$\mathfrak{a} := \frac{n}{n+2} \cdot \inf \left\{ t \in \mathbb{R} : t \cdot L + K_{\tilde{X}_n} \in \text{Pic}(\tilde{X}_n) \text{ is effective} \right\} = \frac{2n}{n+2}$$

and $\mathfrak{b} = 1$ is the codimension of the minimal face of the effective cone of \tilde{X}_n containing $\mathfrak{a} \cdot L + K_{\tilde{X}_n}$.

Next, we compute the prediction of [BT98b] for \mathfrak{c} . The divisor

$$\frac{n+2}{n} \cdot \mathfrak{a} \cdot L + K_{\tilde{X}_n} = (n-2, 0)$$

is not rigid, hence, according to [BT98b, Remark 2.4.4], we consider the natural fibration

$$\phi: \tilde{X}_n \rightarrow P_n := \text{Proj} \left(\bigoplus_{\nu \geq 0} \Gamma(\tilde{X}_n, \mathcal{O}_{\tilde{X}_n}(n-2, 0)^{\otimes \nu}) \right),$$

where we have an isomorphism $\mathbb{P}_{\mathbb{Q}}^2 \cong P_n$ such that

$$\phi: \tilde{X}_n \rightarrow \mathbb{P}_{\mathbb{Q}}^2, \quad (a : b : c : d : z : w) \mapsto (a : c : z).$$

As we have $\phi^{-1}(x) \subseteq \mathbb{V}(w)$ if and only if $x \in \mathbb{V}(a, c)$, we only consider points $x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbb{V}(a, c)$ and determine the predicted asymptotic formula

$$N_{\phi^{-1}(x), \pi^* H}(B) \sim \mathfrak{c}_x B^{\mathfrak{a}_x} (\log B)^{\mathfrak{b}_x-1}$$

for the fiber $\phi^{-1}(x)$. We have isomorphisms

$$\begin{aligned} \mathbb{P}_{\mathbb{Q}}^1 &\rightarrow \phi^{-1}(x), & (b : w) &\mapsto \left(a : b : c : \frac{bc+z^{n+1}w}{a} : z : w \right), & \text{for } a \neq 0, \\ \mathbb{P}_{\mathbb{Q}}^1 &\rightarrow \phi^{-1}(x), & (d : w) &\mapsto \left(a : \frac{ad-z^{n+1}w}{c} : c : d : z : w \right), & \text{for } c \neq 0, \end{aligned}$$

which depend on the choice of $a, c, z \in \mathbb{Q}$ such that $x = (a : c : z)$. We now see that $\pi^* H^{2n/(n+2)}$ restricted to $\phi^{-1}(x)$ is an anticanonical height on $\mathbb{P}_{\mathbb{Q}}^1$, which means that the predicted asymptotic formula is

$$N_{\phi^{-1}(x), \pi^* H}(B^{\frac{n+2}{2n}}) \sim \frac{1}{2} \omega_{\infty, x} \left(\prod_{p \text{ prime}} \lambda_p \omega_{p, x} \right) B,$$

where $\lambda_p = 1 - p^{-1}$. Now, consider the charts

$$\begin{aligned} \mathbb{A}_{\mathbb{Q}}^1 &\rightarrow \phi^{-1}(x), & b &\mapsto \left(a : b : c : \frac{bc+z^{n+1}}{a} : z : 1\right), & \text{for } a \neq 0, \\ \mathbb{A}_{\mathbb{Q}}^1 &\rightarrow \phi^{-1}(x), & d &\mapsto \left(a : \frac{ad-z^{n+1}}{c} : c : d : z : 1\right), & \text{for } c \neq 0. \end{aligned}$$

According to [Pey95, 2.2.1], we have

$$\omega_{\nu,x} := \begin{cases} \int_{\mathbb{Q}_{\nu}} \frac{1}{|a| \max |\mathcal{M}_n(a, b, c, (bc+z^{n+1})/a, z, 1)|_{\nu}^2} db & \text{for } a \neq 0, \\ \int_{\mathbb{Q}_{\nu}} \frac{1}{|c| \max |\mathcal{M}_n(a, (ad-z^{n+1})/c, c, d, z, 1)|_{\nu}^2} dd & \text{for } c \neq 0 \end{cases}$$

for $\nu := p$ and $\nu := \infty$, where we have used the isomorphism

$$\omega_{\phi^{-1}(x)} \cong \mathcal{O}_{\tilde{X}_n}(-2L)|_{\phi^{-1}(x)}$$

identifying the section db from the first chart (resp. the section dd from the second chart) with the section a/w^2 from the Cox ring (resp. the section c/w^2 from the Cox ring). Imposing the conditions $a, c, z \in \mathbb{Z}$ and $\gcd(a, c, z) = 1$, the integrals $\omega_{\nu,x}$ only depend on $x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbb{V}(a, c)$.

It follows that we have $\mathfrak{a}_x = \frac{2n}{n+2} = \mathfrak{a}$, $\mathfrak{b}_x = 1 = \mathfrak{b}$, and

$$\mathfrak{c}_x = \frac{1}{2} \omega_{\infty,x} \prod_{p \text{ prime}} \lambda_p \omega_{p,x}.$$

Summing over all the fibers, we obtain the expected constant

$$\mathfrak{c} = \sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbb{V}(a, c)} \mathfrak{c}_x.$$

in the asymptotic formula for $N_{U,H}(B)$. We show in Corollary 6.4 that this sum converges.

Lemma 6.1. *For every $x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbb{V}(a, c)$ we have*

$$\prod_{p \text{ prime}} \lambda_p \omega_{p,x} = \frac{1}{\zeta(2)}.$$

Proof. A straightforward calculation of the p -adic integrals yields

$$\omega_{p,x} = 1 + \frac{1}{p},$$

from which the result follows. \square

Lemma 6.2. *We have*

$$\omega_{\infty,x} = \begin{cases} \iint \frac{1}{|a|} db dw & \text{for } a \neq 0, \\ \iint \frac{1}{|c|} dd dw & \text{for } c \neq 0. \end{cases}$$

Proof. We consider the case $a \neq 0$ (the case $c \neq 0$ is similar). In our expression for $\omega_{\nu,x}$ above, with $\nu = \infty$, we introduce an additional integration over w as in Lemma 5.2 and obtain

$$\omega_{\infty,x} = \frac{1}{2} \iint \frac{1}{|aw^2|} db dw.$$

Now the transformations $w \mapsto \frac{1}{w^2}$ and then $b \mapsto \frac{b}{w}$ give the result. \square

The following result proves Theorem 1.4.

Theorem 6.3. *Let $\overline{H}: \mathbb{P}^2(\mathbb{Q}) \rightarrow \mathbb{R}_{>0}$ be a height relative to*

$$\mathcal{O}_{\mathbb{P}^2_{\mathbb{Q}}}(-n-1) \cong \mathcal{O}_{P_n}(-1) \otimes \omega_{P_n}.$$

There exist positive constants c_1, c_2 such that for every $x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbb{V}(a, c)$ we have

$$c_1 \overline{H}(x) \leq \mathfrak{c}_x \leq c_2 \overline{H}(x).$$

Proof. Let $x := (a : c : z) \in \mathbb{P}^2(\mathbb{Q})$ with integral coordinates and $\gcd(a, c, z) = 1$. Without loss of generality, we can define the height \overline{H} as

$$\overline{H}(x) := \frac{1}{\max\{|a|, |c|, |z|\}^{n+1}}.$$

First, assume $\max\{|a|, |c|, |z|\} = |a|$. Then we have

$$\omega_{\infty, x} \leq \iint_{|a^n w|, |b| \leq 1} \frac{1}{|a|} db dw = \int_{|a^n w| \leq 1} \frac{2}{|a|} dw \ll \frac{1}{|a|^{n+1}}.$$

Moreover, the conditions $|a^n w|, |b| \leq \frac{1}{2}$ imply all the conditions on the integral $\omega_{\infty, x}$, so that we obtain

$$\omega_{\infty, x} \geq \iint_{|a^n w|, |b| \leq \frac{1}{2}} \frac{1}{|a|} db dw = \frac{1}{|a|^{n+1}}.$$

The case $\max\{|a|, |c|, |z|\} = |c|$ is similar. It remains to consider $\max\{|a|, |c|, |z|\} = |z|$. Assume $|a| \geq |c|$. We have

$$\omega_{\infty, x} \leq \iint_{|b|, |(bc+zn+1w)/a| \leq 1} \frac{1}{|a|} db dw = \int_{|b| \leq 1} \frac{2|a|}{|az^{n+1}|} db \ll \frac{1}{|z|^{n+1}}.$$

Moreover, the conditions $|b| \leq \frac{|a|}{2|c|}$ and $|w| \leq \frac{|a|}{2|z|^{n+1}}$ imply all the conditions on the integral $\omega_{\infty, x}$, so that we obtain

$$\omega_{\infty, x} \geq \iint_{|b| \leq \frac{|a|}{2|c|}, |w| \leq \frac{|a|}{2|z|^{n+1}}} \frac{1}{|a|} db dw = \frac{|a|}{|cz^{n+1}|} \geq \frac{1}{|z|^{n+1}}.$$

The case $|c| \geq |a|$ is similar. Together, we obtain $\overline{H}(x) \asymp \omega_{\infty, x} \asymp \mathfrak{c}_x$. \square

Corollary 6.4. *We have*

$$\sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbb{V}(a, c)} \mathfrak{c}_x < \infty.$$

Proof. Since $n \geq 3$, we have

$$\sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbb{V}(a, c)} \mathfrak{c}_x \ll \sum_{a, c, z} \frac{1}{\max\{|a|, |c|, |z|\}^{n+1}} \ll 1. \quad \square$$

7. THE EXPECTED FORMULA FOR X'_n IN THE CASE $n \geq 2$

The aim of this section is to determine, for $n \geq 2$, the expected asymptotic formula for $N_{U', H'}(B)$ where

$$U' := \tilde{X}'_n \setminus \mathbb{V}(yztw) = X'_n \setminus \mathbb{V}(yzt)$$

and, moreover, to prove Theorem 1.6.

Recall from Section 4 that we consider

$$L' := (n+1) \cdot \pi^*(-K_{X'_n}) = (n^2 + 3n + 2, n^2 + 2n + 1, n + 1, n + 3)$$

in $\text{Pic}(\tilde{X}'_n) \cong \mathbb{Z}^4$, and that the pullback of $(H')^{n+1}$ is a height relative to L' . According to [BM90, Conjecture C'] (see also [Pey03, 3.6]), the predicted asymptotic formula is

$$N_{U', H'}(B) \sim \mathfrak{c} B^{\mathfrak{a}} (\log B)^{\mathfrak{b}-1},$$

where

$$\mathfrak{a} := (n+1) \cdot \inf \left\{ t \in \mathbb{R} : t \cdot L' + K_{\tilde{X}'_n} \in \text{Pic}(\tilde{X}'_n) \text{ is effective} \right\} = \frac{2n+2}{n+3}$$

and $\mathfrak{b} = 1$ is the codimension of the minimal face of the effective cone of \tilde{X}'_n containing $\mathfrak{a} \cdot L' + K_{\tilde{X}'_n}$. A prediction for \mathfrak{c} can be found in [BT98b]. The \mathbb{Q} -divisor

$$\frac{1}{n+1} \cdot \mathfrak{a} \cdot L'_n + K_{\tilde{X}'_n} = \frac{n-1}{n+3} \cdot (n+2, n+1, 1, 0)$$

is not rigid, hence, according to [BT98b, Remark 2.4.4], we consider the natural fibration

$$\phi' : \tilde{X}'_n \dashrightarrow P'_n := \text{Proj} \left(\bigoplus_{\substack{\nu \geq 0 \\ n+3 \mid \nu}} \Gamma(\tilde{X}'_n, \mathcal{O}_{\tilde{X}'_n}(n+2, n+1, 1, 0)^{\otimes \frac{n-1}{n+3} \cdot \nu}) \right),$$

where we have an isomorphism $\mathbb{P}^2_{\mathbb{Q}} \cong P'_n$ such that ϕ' extends to

$$\phi' : \tilde{X}'_n \rightarrow \mathbb{P}^2_{\mathbb{Q}}, \quad (a : b : c : d : y : z : t : w) \mapsto (\hat{a} : \hat{c} : \hat{y}) := (at : ct : yz).$$

As we have $\phi'^{-1}(x) \subseteq \mathbb{V}(yztw)$ if and only if $x \in \mathbb{V}(\hat{a}, \hat{c}) \cup \mathbb{V}(\hat{y})$, we only consider points $x \in \mathbb{P}^2(\mathbb{Q}) \setminus (\mathbb{V}(\hat{a}, \hat{c}) \cup \mathbb{V}(\hat{y}))$ and determine the predicted asymptotic formula

$$N_{\phi'^{-1}(x), \pi^* H'}(B) \sim \mathfrak{c}_x B^{\mathfrak{a}_x} (\log B)^{\mathfrak{b}_x-1}$$

for the fiber $\phi'^{-1}(x)$. We have isomorphisms

$$\begin{aligned} \mathbb{P}^1_{\mathbb{Q}} &\rightarrow \phi'^{-1}(x), & (b : w) &\mapsto \left(\hat{a} : b : \hat{c} : \frac{b\hat{c} + \hat{y}^n w}{\hat{a}} : \hat{y} : 1 : 1 : w \right), & \text{for } \hat{a} \neq 0, \\ \mathbb{P}^1_{\mathbb{Q}} &\rightarrow \phi'^{-1}(x), & (d : w) &\mapsto \left(\hat{a} : \frac{\hat{a}d - \hat{y}^n w}{\hat{c}} : \hat{c} : d : \hat{y} : 1 : 1 : w \right), & \text{for } \hat{c} \neq 0, \end{aligned}$$

which depend on the choice of $\hat{a}, \hat{c}, \hat{y} \in \mathbb{Q}$ such that $x = (\hat{a} : \hat{c} : \hat{y})$. We now see that $(\pi^* H')^{(2n+2)/(n+3)}$ restricted to $\phi'^{-1}(x)$ is an anticanonical height on $\mathbb{P}^1_{\mathbb{Q}}$, which means that the predicted asymptotic formula is

$$N_{\phi'^{-1}(x), \pi^* H'}(B^{\frac{n+3}{2n+2}}) \sim \frac{1}{2} \omega_{\infty, x} \left(\prod_{p \text{ prime}} \lambda_p \omega_{p, x} \right) B,$$

where $\lambda_p = 1 - p^{-1}$. Now, consider the charts

$$\begin{aligned} \mathbb{A}^1_{\mathbb{Q}} &\rightarrow \phi'^{-1}(x), & b &\mapsto \left(\hat{a} : b : \hat{c} : \frac{b\hat{c} + \hat{y}^n}{\hat{a}} : \hat{y} : 1 : 1 : 1 \right), & \text{for } \hat{a} \neq 0, \\ \mathbb{A}^1_{\mathbb{Q}} &\rightarrow \phi'^{-1}(x), & d &\mapsto \left(\hat{a} : \frac{\hat{a}d - \hat{y}^n}{\hat{c}} : \hat{c} : d : \hat{y} : 1 : 1 : 1 \right), & \text{for } \hat{c} \neq 0. \end{aligned}$$

According to [Pey95, 2.2.1], we have

$$\omega_{\nu, x} := \begin{cases} \int_{\mathbb{Q}_{\nu}} \frac{1}{|\hat{a}| \max |\mathcal{M}'_n(\hat{a}, b, \hat{c}, (b\hat{c} + \hat{y}^n)/\hat{a}, \hat{y}, 1, 1, 1)|_{\nu}^{2/(n+3)}} db & \text{for } \hat{a} \neq 0, \\ \int_{\mathbb{Q}_{\nu}} \frac{1}{|\hat{c}| \max |\mathcal{M}'_n(\hat{a}, (\hat{a}d - \hat{y}^n)/\hat{c}, \hat{c}, d, \hat{y}, 1, 1, 1)|_{\nu}^{2/(n+3)}} dd & \text{for } \hat{c} \neq 0 \end{cases}$$

for $\nu := p$ and $\nu := \infty$, where we have used the isomorphism

$$\omega_{\phi'^{-1}(x)}^{\otimes (n+3)} \cong \mathcal{O}_{\tilde{X}'_n}(-2L'_n)|_{\phi'^{-1}(x)}$$

identifying the section $(db)^{\otimes(n+3)}$ from the first chart (resp. the section $(dd)^{\otimes(n+3)}$ from the second chart) with the section a^{n+3}/w^{2n+6} from the Cox ring (resp. the section c^{n+3}/w^{2n+6} from the Cox ring). Imposing the conditions $\hat{a}, \hat{c}, \hat{y} \in \mathbb{Z}$ and $\gcd(\hat{a}, \hat{c}, \hat{y}) = 1$, the integrals $\omega_{\nu, x}$ only depend on $x \in \mathbb{P}^2(\mathbb{Q}) \setminus (\mathbb{V}(\hat{a}, \hat{c}) \cup \mathbb{V}(\hat{y}))$.

It follows that we have $\mathbf{a}_x = \frac{2n}{n+2} = \mathbf{a}$, $\mathbf{b}_x = 1 = \mathbf{b}$, and

$$\mathbf{c}_x = \frac{1}{2} \omega_{\infty, x} \prod_{p \text{ prime}} \lambda_p \omega_{p, x}.$$

Summing over all the fibers, we obtain the expected constant

$$\mathbf{c} = \sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus (\mathbb{V}(\hat{a}, \hat{c}) \cup \mathbb{V}(\hat{y}))} \mathbf{c}_x.$$

in the asymptotic formula for $N_{U', H'}(B)$. We show in Proposition 9.3 that $\mathbf{c} < \infty$.

Lemma 7.1. *Let $\mathfrak{e} := \frac{-n+1}{n+3}$. For every $x \in \mathbb{P}^2(\mathbb{Q}) \setminus (\mathbb{V}(\hat{a}, \hat{c}) \cup \mathbb{V}(\hat{y}))$ we have*

$$\omega_{p, x} = \left(\left(1 - \frac{1}{p}\right) \cdot \frac{1 - (p^{\mathfrak{e}})^{\nu_p(\hat{y})+1}}{1 - p^{\mathfrak{e}}} + \frac{1}{p} + \frac{(p^{\mathfrak{e}})^{\nu_p(\hat{y})}}{p} \right) \cdot (p^{\mathfrak{e}+1})^{\min(\nu_p(\hat{a}), \nu_p(\hat{c}))}.$$

Proof. A lengthy, but straightforward calculation of the p -adic integrals yields

$$\omega_{p, x} = \begin{cases} 1 + \left(1 - \frac{1}{p}\right) \sum_{j=1}^{\nu_p(\hat{y})-1} (p^{\mathfrak{e}})^j + (p^{\mathfrak{e}})^{\nu_p(\hat{y})} & \text{for } \nu_p(\hat{y}) > 0, \\ \left(1 + \frac{1}{p}\right) \cdot (p^{\mathfrak{e}+1})^{\min(\nu_p(\hat{a}), \nu_p(\hat{c}))} & \text{otherwise,} \end{cases}$$

from which the result follows. \square

Lemma 7.2. *We have*

$$\omega_{\infty, x} = \begin{cases} \iint_{\max |\mathcal{M}'_n(\hat{a}, b, \hat{c}, (b\hat{c} + \hat{y}^n/\hat{a}), \hat{y}, 1, 1, w)| \leq 1} \frac{1}{|\hat{a}|} db dw & \text{for } \hat{a} \neq 0, \\ \iint_{\max |\mathcal{M}'_n(\hat{a}, (\hat{a}d - \hat{y}^n)/\hat{c}, \hat{c}, d, \hat{y}, 1, 1, w)| \leq 1} \frac{1}{|\hat{c}|} db dw & \text{for } \hat{c} \neq 0. \end{cases}$$

Proof. This is completely analogous to Lemma 6.2. \square

The following result proves Theorem 1.6.

Theorem 7.3. *Let $\overline{H}: \mathbb{P}^2(\mathbb{Q}) \rightarrow \mathbb{R}_{>0}$ be a height relative to an arbitrary line bundle. There does not exist an open subset $V \subseteq \mathbb{P}^2(\mathbb{Q})$ with positive constants c_1, c_2 such that for every $x \in V$ we have*

$$c_1 \overline{H}(x) \leq \mathbf{c}_x \leq c_2 \overline{H}(x).$$

Proof. Let $x := (\hat{a} : \hat{c} : \hat{y}) \in \mathbb{P}^2(\mathbb{Q})$ with integral coordinates and $\gcd(\hat{a}, \hat{c}, \hat{y}) = 1$. Without loss of generality, we can define the height \overline{H} as $\overline{H}(x) := \max\{|a|, |c|, |z|\}^r$ for some $r \in \mathbb{Z}$. We define

$$\omega_{\infty, x}^- := \int_{\mathbb{R}} \frac{1}{|\hat{a}| \max |\mathcal{M}'_n(\hat{a}, 2b, \hat{c}, 0, \hat{y}, 1, 1, 1)|_{\infty}^{2/(n+3)}} db,$$

$$\omega_{\infty, x}^+ := \int_{\mathbb{R}} \frac{1}{|\hat{a}| \max |\mathcal{M}'_n(\hat{a}, b, \hat{c}, 0, \hat{y}, 1, 1, 1)|_{\infty}^{2/(n+3)}} db.$$

For $|\hat{a}| \geq |\hat{c}|$ and $|\hat{a}| \gg_{\hat{y}} 1$, we have $\omega_{\infty, x}^- \leq \omega_{\infty, x}$. We also have $\omega_{\infty, x} \leq \omega_{\infty, x}^+ = 2\omega_{\infty, x}^-$. Now let

$$x_0(m) := (\hat{a}_0 p^m : \hat{c}_0 : \hat{y}_0),$$

$$x'_0(m) := (\hat{a}_0 p^m : \hat{c}_0 p^m : \hat{y}_0),$$

where $\hat{a}_0, \hat{c}_0, \hat{y}_0 \in \mathbb{Z}$, $\gcd(\hat{a}_0, \hat{c}_0, \hat{y}_0) = 1$, and p is prime with $p \nmid \hat{a}_0, \hat{c}_0, \hat{y}_0$. For $m \rightarrow \infty$, we have

$$\mathfrak{c}_{x_0(m)} \asymp \omega_{\infty, x_0(m)} \asymp \omega_{\infty, x_0(m)}^- \asymp \omega_{\infty, x'_0(m)}^-,$$

since $\omega_{\infty, x}^-$ does not depend on \hat{c} for $|\hat{a}| \geq |\hat{c}|$. We also have

$$\mathfrak{c}_{x'_0(m)} \asymp (p^m)^{\mathfrak{c}+1} \omega_{\infty, x'_0(m)} \asymp (p^m)^{\mathfrak{c}+1} \omega_{\infty, x'_0(m)}^- \asymp (p^m)^{\mathfrak{c}+1} \mathfrak{c}_{x_0(m)},$$

but $x_0(m)$ and $x'_0(m)$ have the same height. \square

8. ESTIMATING INTEGRAL POINTS ON THE UNIVERSAL TORSOR OF X_n

We are going to prove Theorem 1.1 by showing

$$N_{U,H}(B^2) = \mathfrak{c}B^2 \log(B^2) + O(B^2),$$

where \mathfrak{c} is as in Section 5, and we are going to prove Theorem 1.2 by showing

$$N_{U,H}(B^{\frac{n+2}{n}}) = \mathfrak{c}B^2 + O(B^{\frac{n+2}{n}})$$

for $n \geq 3$, where \mathfrak{c} is as in Section 6.

We use [Der09, Lemma 3.1] and [DF14, Lemma 3.6] repeatedly to approximate sums by integrals. Note that we have

$$\max |\mathcal{M}_n(a, b, c, d, z, w, t)| = \max\{|b|, |d|, |a^n w t^{n+1}|, |c^n w t^{n+1}|, |z^n w t|\}.$$

We define

$$V_1(a, c, z, w; B) := \begin{cases} \int_{|d|, |(ad - z^{n+1}w)/c| \leq B} \frac{1}{|c|} dd & \text{for } c \neq 0, \\ \int_{|b|, |(bc + z^{n+1}w)/a| \leq B} \frac{1}{|a|} db & \text{for } a \neq 0. \end{cases}$$

Note that for $ac \neq 0$ the two cases coincide. Moreover, we have

$$V_1(a, c, z, w; B) \leq \frac{B}{\max\{|a|, |c|\}}.$$

We also define

$$V_2(a, c, z; B) := \int_{\substack{1 \leq |w| \leq B \\ |a^n w| \leq B \\ |c^n w| \leq B \\ |z^n w| \leq B}} V_1(a, c, z, w; B) dw.$$

Proposition 8.1. *For $n \geq 2$, we have*

$$N_{U,H}(B^{\frac{n+2}{n}}) = \frac{1}{4\zeta(2)} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ \gcd(a, c, z) = 1 \\ (a, c) \neq (0, 0)}} V_2(a, c, z; B) + O(B^{\frac{n+2}{n}}).$$

Proof. By Corollary 4.3, we have

$$N_{U,H}(B^{\frac{n+2}{n}}) = \frac{1}{8} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |w|, |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1 \\ |a^n w t^{n+1}|, |c^n w t^{n+1}|, |z^n w t| \leq B}} \# \left\{ (b, d) \in \mathbb{Z}^2 : \begin{array}{l} ad - bc = z^{n+1}w \\ |b|, |d| \leq B \\ \gcd(b, d, w) = 1 \end{array} \right\}.$$

We apply a Möbius inversion to the condition $\gcd(b, d, w) = 1$ and obtain

$$\begin{aligned}
N_{U,H}(B^{\frac{n+2}{n}}) &= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |w|, |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1 \\ |\alpha a^n w t^{n+1}|, |\alpha c^n w t^{n+1}|, |\alpha z^n w t| \leq B}} \# \left\{ (b, d) \in \mathbb{Z}^2 : \begin{array}{l} ad - bc = z^{n+1} w \\ |b|, |d| \leq B/\alpha \end{array} \right\} \\
&= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |w|, |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1 \\ |\alpha a^n w t^{n+1}|, |\alpha c^n w t^{n+1}|, |\alpha z^n w t| \leq B}} (V_1(a, c, z, w; B/\alpha) + O(1)) \\
&= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |w|, |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1 \\ |\alpha a^n w t^{n+1}|, |\alpha c^n w t^{n+1}|, |\alpha z^n w t| \leq B}} \left(\frac{1}{\alpha} V_1(a, c, z, \alpha w; B) + O(1) \right) \\
&= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8\alpha} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |w|, |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1 \\ |\alpha a^n w t^{n+1}|, |\alpha c^n w t^{n+1}|, |\alpha z^n w t| \leq B}} V_1(a, c, z, \alpha w; B) + O(B^{\frac{n+2}{n}}),
\end{aligned}$$

where the condition $\gcd(a, c) = 1$ is used for the second equality, the transformation $\alpha b \mapsto b$ or $\alpha d \mapsto d$ is applied inside the integral V_1 for the third equality, and the fourth equality follows from the estimate

$$\begin{aligned}
\sum_{\alpha} \sum_{a, c, z, w, t} 1 &\ll \sum_{\alpha} \sum_{a, c, z, t} \frac{B}{\alpha \max\{|a|, |c|\}^n |t|^{n+1}} \\
&\ll \sum_{\alpha} \sum_{a, c, t} \frac{B^{(n+1)/n}}{\alpha^{(n+1)/n} \max\{|a|, |c|\}^n |t|^{n+2/n}} \\
&\ll \begin{cases} B^{(n+1)/n} \log B & \text{for } n = 2 \\ B^{(n+1)/n} & \text{for } n \geq 3 \end{cases} \\
&\ll B^{\frac{n+2}{n}}.
\end{aligned}$$

Replacing the sum over w by an integral, we obtain that $N_{U,H}(B^{\frac{n+2}{n}})$ is equal to

$$\begin{aligned}
&\sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8\alpha} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1 \\ |\alpha a^n t^{n+1}| \leq B \\ |\alpha c^n t^{n+1}| \leq B \\ |\alpha z^n t| \leq B}} \left(\int_{\substack{|w| \geq 1 \\ |\alpha a^n w t^{n+1}| \leq B \\ |\alpha c^n w t^{n+1}| \leq B \\ |\alpha z^n w t| \leq B}} V_1(a, c, z, \alpha w; B) dw + O(R_1) \right) + O(B^{\frac{n+2}{n}}) \\
&= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8\alpha} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1 \\ |\alpha a^n t^{n+1}| \leq B \\ |\alpha c^n t^{n+1}| \leq B \\ |\alpha z^n t| \leq B}} \left(\frac{1}{\alpha} \int_{\substack{|w| \geq \alpha \\ |a^n w t^{n+1}| \leq B \\ |c^n w t^{n+1}| \leq B \\ |z^n w t| \leq B}} V_1(a, c, z, w; B) dw + O(R_1) \right) + O(B^{\frac{n+2}{n}})
\end{aligned}$$

$$= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8\alpha^2} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1}} \int_{\substack{|w| \geq \alpha \\ |a^n w t^{n+1}| \leq B \\ |c^n w t^{n+1}| \leq B \\ |z^n w t| \leq B}} V_1(a, c, z, w; B) dw + O(B^{\frac{n+2}{n}}),$$

where we have applied the transformation $\alpha w \mapsto w$ for the first equality and the second equality follows from the estimates

$$R_1 = \max_w V_1(a, c, z, w; B) \leq \frac{B}{\max\{|a|, |c|\}}$$

and

$$\begin{aligned} \sum_{\alpha} \frac{1}{\alpha} \sum_{a, c, z, t} \frac{B}{\max\{|a|, |c|\}} &\ll \sum_{\alpha} \frac{1}{\alpha} \sum_{a, c, t} \frac{B^{(n+1)/n}}{\alpha^{1/n} \max\{|a|, |c|\} |t|^{1/n}} \\ &\ll \sum_{\alpha} \frac{1}{\alpha} \sum_t \frac{B^{(n+2)/n}}{\alpha^{2/n} |t|^{(n+2)/n}} \ll B^{\frac{n+2}{n}}. \end{aligned}$$

Next, we replace the condition $\alpha \leq |w|$ by the condition $|t^{-1}| \leq |w|$. For $0 < \epsilon < \frac{1}{n}$, we have

$$\begin{aligned} &\sum_{\alpha} \frac{1}{\alpha^2} \sum_{a, c, z, t} \int_{\substack{|t^{-1}| \leq |w| \leq \alpha \\ |a^n w t^{n+1}| \leq B \\ |c^n w t^{n+1}| \leq B \\ |z^n w t| \leq B}} V_1(a, c, z, w; B) dw \\ &\ll \sum_{\alpha} \frac{1}{\alpha^2} \sum_{a, c, z, t} \int_{\substack{|t^{-1}| \leq |w| \leq \alpha \\ |a^n w t^{n+1}| \leq B \\ |c^n w t^{n+1}| \leq B \\ |z^n w t| \leq B}} \frac{B}{\max\{|a|, |c|\}} dw \\ &\ll \sum_{\alpha} \frac{1}{\alpha^2} \sum_{\substack{a, c, z, t \\ z \neq 0 \\ |at|^n, |ct|^n, |z|^n \leq B}} \frac{\alpha^{1-\epsilon} B^{1+\epsilon}}{\max\{|a|, |c|\} |z|^{\epsilon n} |t|^{\epsilon}} + \sum_{\alpha} \frac{1}{\alpha^2} \sum_{\substack{a, c, t \\ z=0 \\ |at|^n, |ct|^n \leq B}} \frac{\alpha^{1-\epsilon} B^{1+\epsilon}}{\max\{|a|, |c|\}^{1+\epsilon n} |t|^{\epsilon(n+1)}} \\ &\ll \sum_{\alpha} \frac{1}{\alpha^2} \sum_{\substack{a, c, t \\ |at|^n, |ct|^n \leq B}} \frac{\alpha^{1-\epsilon} B^{(n+1)/n}}{\max\{|a|, |c|\} |t|^{\epsilon}} \ll \sum_{\alpha} \frac{1}{\alpha^{1+\epsilon}} \sum_t \frac{B^{(n+2)/n}}{|t|^{1+\epsilon}} \ll B^{\frac{n+2}{n}}, \end{aligned}$$

so that we obtain that $N_{U, H}(B^{\frac{n+2}{n}})$ is equal to

$$\begin{aligned} &\frac{1}{8\zeta(2)} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1}} \int_{\substack{|wt| \geq 1 \\ |a^n w t^{n+1}| \leq B \\ |c^n w t^{n+1}| \leq B \\ |z^n w t| \leq B}} V_1(a, c, z, w; B) dw + O(B^{\frac{n+2}{n}}) \\ &= \frac{1}{8\zeta(2)} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1}} \int_{\substack{|(at)^n w| \leq B \\ |(ct)^n w| \leq B \\ |z^n w| \leq B}} \frac{1}{|t|} V_1(a, c, z, w/t; B) dw + O(B^{\frac{n+2}{n}}) \\ &= \frac{1}{8\zeta(2)} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ |t| \geq 1 \\ \gcd(a, c) = \gcd(z, t) = 1}} \int_{\substack{|w| \geq 1 \\ |(at)^n w| \leq B \\ |(ct)^n w| \leq B \\ |z^n w| \leq B}} V_1(ta, tc, z, w; B) dw + O(B^{\frac{n+2}{n}}) \\ &= \frac{1}{4\zeta(2)} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ \gcd(a, c, z) = 1 \\ (a, c) \neq (0, 0)}} \int_{\substack{|w| \geq 1 \\ |a^n w| \leq B \\ |c^n w| \leq B \\ |z^n w| \leq B}} V_1(a, c, z, w; B) dw + O(B^{\frac{n+2}{n}}), \end{aligned}$$

where the transformation $tw \mapsto w$ is applied for the first equality and the 2-to-1 substitution $(ta, tc) \mapsto (a, c)$ is applied for the third equality. Finally, adding the condition $|w| \leq B$ leaves the integral unchanged. \square

We define

$$V_2'(a, c, z) := \int_{\substack{|a^n w| \leq 1 \\ |c^n w| \leq 1 \\ |z^n w| \leq 1}} V_1(a, c, z, w; B) dw.$$

Corollary 8.2. *For $n \geq 2$, we have*

$$N_{U,H}(B^{\frac{n+2}{n}}) = \frac{1}{4\zeta(2)} \sum_{\substack{|a|, |c|, |z| \leq B^{1/n} \\ \gcd(a, c, z) = 1 \\ (a, c) \neq (0, 0)}} V_2'(a, c, z) B^2 + O(B^{\frac{n+2}{n}}).$$

Proof. In the formula from Proposition 8.1, we may restrict the sum to $|a|, |c|, |z| \leq B^{1/n}$ since $V_2(a, c, z; B)$ vanishes otherwise. The transformations $(b, w) \mapsto (Bb, Bw)$ for $a \neq 0$ and $(d, w) \mapsto (Bd, Bw)$ for $c \neq 0$ show that we have

$$\int_{\substack{|a^n w| \leq B \\ |c^n w| \leq B \\ |z^n w| \leq B}} V_1(a, c, z, w; B) dw = V_2'(a, c, z) B^2.$$

Comparing the left side with $V_2(a, c, z; B)$, we see that the condition $|w| \leq B$ in the definition of $V_2(a, c, z; B)$ follows from the other conditions, and it remains to remove the condition $|w| \geq 1$. The corollary now follows from the computation

$$\sum_{|a|, |c|, |z| \leq B^{1/n}} \int_{|w| \leq 1} V_1(a, c, z, w; B) dw \ll \sum_{|a|, |c|, |z| \leq B^{1/n}} \frac{B}{\max\{|a|, |c|\}} \ll B^{\frac{n+2}{n}}. \quad \square$$

Remark 8.3. For $n \geq 3$, we have $V_2'(a, c, z) = \omega_{\infty, (a:c:z)}$ according to Lemma 6.2 and hence

$$V_2'(a, c, z) \asymp \frac{1}{\max\{|a|, |c|, |z|\}^{n+1}},$$

but this also holds for $n = 2$. Theorem 6.3 and Corollary 8.2 now yield

$$N_{U,H}(B^{\frac{n+2}{n}}) \asymp \sum_{\substack{|a|, |c|, |z| \leq B^{1/n} \\ \gcd(a, c, z) = 1 \\ (a, c) \neq (0, 0)}} \frac{B^2}{\max\{|a|, |c|, |z|\}^{n+1}} \asymp \begin{cases} B^2 \log B, & n = 2, \\ B^2, & n \geq 3. \end{cases}$$

In the following, we turn these upper and lower bounds into asymptotic formulas.

We begin with the case $n \geq 3$.

Theorem 8.4. *For $n \geq 3$, we have*

$$N_{U,H}(B^{\frac{n+2}{n}}) = \mathfrak{c} B^2 + O(B^{\frac{n+2}{n}}).$$

Proof. We remove the conditions $|a|, |c|, |z| \leq B^{1/n}$ from the sum in Corollary 8.2 with a satisfactory error term. Since $\omega_{\infty, (a:c:z)} \ll \max\{|a|, |c|, |z|\}^{-n-1}$ by Theorem 6.3, we have

$$\sum_{\max\{|a|, |c|, |z|\} > B^{1/n}} \omega_{\infty, (a:c:z)} B^2 \ll \sum_{\substack{|a| \leq |c| \leq |z| \\ |z| > B^{1/n}}} \frac{B^2}{|z|^{n+1}} \ll \sum_{|z| > B^{1/n}} \frac{B^2}{|z|^{n-1}} \ll B^{\frac{n+2}{n}}.$$

It follows that we have

$$\begin{aligned} N_{U,H}(B^{\frac{n+2}{n}}) &= \frac{1}{4\zeta(2)} \left(\sum_{\substack{|a|,|c|,|z|\geq 0 \\ \gcd(a,c,z)=1 \\ (a,c)\neq(0,0)}} \omega_{\infty,(a:c:z)} \right) B^2 + O(B^{\frac{n+2}{n}}) \\ &= \left(\sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbb{V}(a,c)} \frac{\omega_{\infty,x}}{2\zeta(2)} \right) B^2 + O(B^{\frac{n+2}{n}}), \end{aligned}$$

as predicted in Section 6. \square

We now turn to the case $n = 2$. Note that we already have a result on the order of magnitude in Remark 8.3 following from Corollary 8.2. In order to obtain an asymptotic formula, we resume our calculation from Proposition 8.1. We define

$$V_3(B) := \iiint V_2(a, c, z; B) \, da \, dc \, dz.$$

Lemma 8.5. *For $n = 2$, we have*

$$V_3(B) = \omega_{\infty} B^2 \log B.$$

Proof. We have

$$V_3(B) = \int \cdots \int_{\substack{1 \leq |w| \leq B \\ |a^2 w|, |c^2 w|, |z^2 w| \leq B \\ |(ad-z^3 w)/c|, |d| \leq B}} \frac{1}{|c|} \, da \, dc \, dd \, dz \, dw.$$

Applying the transformations $(a, c, z) \mapsto B^{1/2}/|w|^{1/2}(a, c, z)$ and $b \mapsto Bb$, we obtain

$$\begin{aligned} V_3(B) &= B^2 \int \cdots \int_{\substack{1 \leq |w| \leq B \\ |a^2|, |c^2|, |z^2| \leq B \\ |(ad-z^3)/c|, |d| \leq B}} \frac{1}{|cw|} \, da \, dc \, dd \, dz \, dw \\ &= B^2 \int_{1 \leq |w| \leq B} \frac{1}{|w|} \, dw \iiint \int_{\substack{|a^2|, |c^2|, |z^2| \leq 1 \\ |(ad-z^3)/c|, |d| \leq 1}} \frac{1}{|c|} \, da \, dc \, dd \, dz. \end{aligned}$$

Now the transformation $|w| \mapsto B^{|w|}$ (with $dw \mapsto B^{|w|} \log B \, dw$) yields

$$\begin{aligned} V_3(B) &= B^2 \log B \int_{0 \leq |w| \leq 1} dw \iiint \int_{\substack{|a^2|, |c^2|, |z^2| \leq 1 \\ |(ad-z^3)/c|, |d| \leq 1}} \frac{1}{|c|} \, da \, dc \, dd \, dz \\ &= 2B^2 \log B \iiint \int_{\substack{|a^2|, |c^2|, |z^2| \leq 1 \\ |(ad-z^3)/c|, |d| \leq 1}} \frac{1}{|c|} \, da \, dc \, dd \, dz \\ &= \omega_{\infty} B^2 \log B, \end{aligned}$$

where the last equality follows from Lemma 5.2. \square

Theorem 8.6. *For $n = 2$, we have*

$$N_{U,H}(B^2) = \epsilon B^2 \log(B^2) + O(B^2).$$

Proof. According to Proposition 8.1, we have

$$N_{U,H}(B^2) = \frac{1}{4\zeta(2)} \sum_{\substack{|a|,|c|,|z|\geq 0 \\ \gcd(a,c,z)=1 \\ (a,c)\neq(0,0)}} V_2(a, c, z; B) + O(B^2).$$

We apply a Möbius inversion to the condition $\gcd(a, c, z) = 1$ and replace the sum over z by an integral to obtain

$$\begin{aligned}
N_{U,H}(B^2) &= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)} \sum_{\substack{|a|, |c|, |z| \geq 0 \\ (a, c) \neq (0, 0)}} V_2(\alpha a, \alpha c, \alpha z; B) + O(B^2) \\
&= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)} \sum_{\substack{|a|, |c| \geq 0 \\ (a, c) \neq (0, 0)}} \left(\int V_2(\alpha a, \alpha c, \alpha z; B) dz + O(R_2) \right) + O(B^2) \\
&= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)\alpha} \sum_{\substack{|a|, |c| \geq 0 \\ (a, c) \neq (0, 0)}} \int V_2(\alpha a, \alpha c, z; B) dz + O(B^2) \\
&= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)\alpha} \sum_{|a|, |c| \geq 1} \int V_2(\alpha a, \alpha c, z; B) dz + O(B^2),
\end{aligned}$$

where the third equality follows from the transformation $\alpha z \mapsto z$ and the estimates

$$R_2 \leq \max_z \int_{\substack{1 \leq |w| \leq B \\ |\alpha^2 a^2 w| \leq B \\ |\alpha^2 c^2 w| \leq B \\ |\alpha^2 z^2 w| \leq B}} \frac{B}{\max\{|\alpha a|, |\alpha c|\}} dw \ll \frac{B^2}{\alpha^3 \max\{|a|, |c|\}^3}$$

and

$$\sum_{\alpha, a, c} \frac{B^2}{\alpha^3 \max\{|a|, |c|\}^3} \ll B^2.$$

Note that for every $a, c \in \mathbb{R}$ with $(a, c) \neq (0, 0)$ we have

$$\begin{aligned}
\int V_2(a, c, z; B) dz &\leq \int \int_{\substack{1 \leq |w| \leq B \\ |a^2 w| \leq B \\ |c^2 w| \leq B \\ |z^2 w| \leq B}} \frac{B}{\max\{|a|, |c|\}} dw dz \\
&\leq \int_{\substack{1 \leq |w| \leq B \\ |a^2 w| \leq B \\ |c^2 w| \leq B}} \frac{B^{3/2}}{\max\{|a|, |c|\} |w|^{1/2}} dw \\
&\leq \min \left\{ \frac{B^2}{\max\{|a|, |c|\}^2}, \frac{B^2}{\max\{|a|, |c|\}} \right\},
\end{aligned}$$

which in particular implies the last equality above. We now successively replace the sums over a and c by integrals to obtain

$$\begin{aligned}
N_{U,H}(B^2) &= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)\alpha} \sum_{|c| \geq 1} \left(\iint_{|a| \geq 1} V_2(\alpha a, \alpha c, z; B) da dz + O(R_3) \right) + O(B^2) \\
&= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)\alpha} \left(\iiint_{|a|, |c| \geq 1} V_2(\alpha a, \alpha c, z; B) da dc dz + O(R_4) \right) + O(B^2) \\
&= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)\alpha^3} \iiint_{|a|, |c| \geq \alpha} V_2(a, c, z; B) da dc dz + O(B^2)
\end{aligned}$$

since we have

$$R_3 = \max_a \int V_2(\alpha a, \alpha c, z; B) dz \ll \frac{B^2}{\alpha^2 |c|^2} \quad \text{and} \quad \sum_{\alpha} \frac{1}{\alpha} \sum_c \frac{B^2}{\alpha^2 |c|^2} \ll B^2$$

as well as

$$R_4 = \max_c \iint_{|a| \geq 1} V_2(\alpha a, \alpha c, z; B) da dz \ll \frac{B^2}{\alpha^2} \quad \text{and} \quad \sum_{\alpha} \frac{1}{\alpha} \cdot \frac{B^2}{\alpha^2} \ll B^2$$

and moreover the transformations $\alpha a \mapsto a$ as well as $\alpha b \mapsto b$ have been applied for the last equality. Finally, we can remove the conditions $|a|, |c| \geq \alpha$ from the integral in order to obtain $V_3(B)$ since

$$\begin{aligned} & \sum_{\alpha} \frac{1}{\alpha^3} \iiint_{|a|, |c| \leq \alpha} V_2(a, c, z; B) da dc dz \\ & \ll \sum_{\alpha} \frac{1}{\alpha^3} \iint_{|a|, |c| \leq \alpha} \frac{B^2}{\max\{|a|, |c|\}} da dc \\ & \ll \sum_{\alpha} \frac{B^2}{\alpha^2} \ll B^2, \end{aligned}$$

as well as

$$\begin{aligned} & \sum_{\alpha} \frac{1}{\alpha^3} \iiint_{|a| \leq \alpha, |c| \geq \alpha} V_2(a, c, z; B) da dc dz \\ & \ll \sum_{\alpha} \frac{1}{\alpha^3} \iint_{|a| \leq \alpha, |c| \geq \alpha} \frac{B^2}{|c|^2} da dc \\ & \ll \sum_{\alpha} \frac{1}{\alpha^3} \int_{|a| \leq \alpha} \frac{B^2}{\alpha} da \ll \sum_{\alpha} \frac{B^2}{\alpha^3} \ll B^2, \end{aligned}$$

The case $|a| \geq \alpha, |c| \leq \alpha$ is handled similarly. It follows that we obtain

$$\begin{aligned} N_{U,H}(B^2) &= \frac{1}{4\zeta(2)} V_3(B) + O(B^2) \\ &= \frac{\omega_{\infty}}{4\zeta(2)\zeta(3)} B^2 \log B + O(B^2) \\ &= \frac{\omega_{\infty}}{8\zeta(2)\zeta(3)} B^2 \log(B^2) + O(B^2), \end{aligned}$$

by Lemma 8.5, as predicted in Section 5. \square

9. ESTIMATING INTEGRAL POINTS ON THE UNIVERSAL TORSOR OF X'_n

We are going to prove Theorem 1.5 by showing

$$N_{U',H'}(B^{\frac{1}{n+1}}) = \mathfrak{c} B^{\frac{2}{n+3}} + O_{\epsilon}(B^{\frac{1}{n+1}+\epsilon})$$

for $n \geq 2$ and any $\epsilon > 0$, where \mathfrak{c} is as in Section 7.

As in the preceding section, we repeatedly use [Der09, Lemma 3.1] and [DF14, Lemma 3.6] to approximate sums by integrals. We define

$$\mathcal{H}(b, d, w, a, c, y, z, t) := \max |\mathcal{M}'_n(a, b, c, d, y, z, t, w)|.$$

Moreover, we define

$$V_{1,\lambda}(a, c, y, z, t, w; B) := \begin{cases} \int_{\mathcal{H}(\lambda b, (\lambda b c + y^n z^{n+1} w)/a, \dots) \leq B} \frac{1}{|a|} db & \text{for } a \neq 0, \\ \int_{\mathcal{H}((\lambda a d - y^n z^{n+1} w)/c, \lambda d, \dots) \leq B} \frac{1}{|c|} dd & \text{for } c \neq 0. \end{cases}$$

Note that for $ac \neq 0$ the two cases coincide and that we have

$$V_{1,1}(a, c, y, z, t, w; B) \ll \frac{B^{1/(n+3)}}{\max\{|a|, |c|\}^{(n+5)/(n+3)} |y|^{2/(n+3)}}.$$

We also define

$$V_2(a, c, y, z, t; B) := \int V_{1,1}(a, c, y, z, t, w; B) dw.$$

Theorem 9.1. *Let $n \geq 2$. For any $\epsilon > 0$, we have*

$$N_{U', H'}(B^{\frac{1}{n+1}}) = \mathbf{c} B^{\frac{2}{n+3}} + O_\epsilon(B^{\frac{1}{n+1} + \epsilon}).$$

Proof. By Corollary 4.7, we have that $16 \cdot N_{U', H'}(B^{\frac{1}{n+1}})$ is equal to

$$\sum_{\substack{|a|, |c| \geq 0 \\ |y|, |z|, |t|, |w| \geq 1 \\ \gcd(a, c) = \gcd(y, w) = 1 \\ \gcd(y, t) = \gcd(z, t) = 1}} \# \left\{ (b, d) \in \mathbb{Z}^2 : \begin{array}{l} ad - bc = y^n z^{n+1} w \\ \mathcal{H}(\dots) \leq B \\ \gcd(b, d, z) = \gcd(b, d, w) = 1 \end{array} \right\}.$$

We apply a Möbius inversion to the conditions

$$\gcd(b, d, z) = \gcd(b, d, w) = 1,$$

so that, after using the transformation $(b, d) \mapsto ([\alpha, \beta]b, [\alpha, \beta]d)$, we obtain that $16 \cdot N_{U', H'}(B^{\frac{1}{n+1}})$ is equal to

$$\begin{aligned} & \sum_{\substack{|a|, |c| \geq 0 \\ |y|, |z|, |t|, |w| \geq 1 \\ \gcd(a, c) = \gcd(y, w) = 1 \\ \gcd(y, t) = \gcd(z, t) = 1}} \sum_{\substack{\alpha, \beta > 0 \\ \alpha | z \\ \beta | w}} \mu(\alpha) \mu(\beta) \# \left\{ (b, d) \in \mathbb{Z}^2 : \begin{array}{l} ad - bc = y^n z^{n+1} w / [\alpha, \beta] \\ \mathcal{H}([\alpha, \beta]b, [\alpha, \beta]d, \dots) \leq B \end{array} \right\} \\ &= \sum_{\substack{|a|, |c| \geq 0 \\ |y|, |z|, |t|, |w| \geq 1 \\ \gcd(a, c) = \gcd(y, w) = 1 \\ \gcd(y, t) = \gcd(z, t) = 1 \\ \mathcal{H}(0, 0, \dots) \leq B}} \sum_{\substack{\alpha, \beta > 0 \\ \alpha | z \\ \beta | w}} \mu(\alpha) \mu(\beta) \left(V_{1, [\alpha, \beta]}(a, c, y, z, t, w; B) + O(1) \right) \\ &= \sum_{\substack{|a|, |c| \geq 0 \\ |y|, |z|, |t|, |w| \geq 1 \\ \gcd(a, c) = \gcd(y, w) = 1 \\ \gcd(y, t) = \gcd(z, t) = 1 \\ \mathcal{H}(0, 0, \dots) \leq B}} \sum_{\substack{\alpha, \beta > 0 \\ \alpha | z \\ \beta | w}} \mu(\alpha) \mu(\beta) \left(\frac{1}{[\alpha, \beta]} V_{1,1}(a, c, y, z, t, w; B) + O(1) \right) \\ &= \sum_{\substack{|a|, |c| \geq 0 \\ |y|, |z|, |t|, |w| \geq 1 \\ \gcd(a, c) = \gcd(y, w) = 1 \\ \gcd(y, t) = \gcd(z, t) = 1}} \sum_{\substack{\alpha, \beta > 0 \\ \alpha | z \\ \beta | w}} \frac{\mu(\alpha) \mu(\beta)}{[\alpha, \beta]} V_{1,1}(a, c, y, z, t, w; B) + O(B^{\frac{1}{n+1} + \epsilon}) \end{aligned}$$

for any $\epsilon > 0$, where we have used the fact that $\mathcal{H}(0, 0, \dots) \leq B$ implies

$$\max\{|a|, |c|\}^{n^2+n} |y|^{n+1} |zt|^{2n+2} |w|^{n+3} \leq B$$

to obtain the estimate

$$\begin{aligned} \sum_{a, c, y, z, t, w} \sum_{\substack{\alpha | z \\ \beta | w}} 1 &\ll \sum_{a, c, y, z, t, w} 2^{\omega(z) + \omega(w)} \\ &\ll \sum_{a, c, y, z, t} 2^{\omega(z)} \frac{B^{1/(n+3)} \log B}{\max\{|a|, |c|\}^{(n^2+n)/(n+3)} |y|^{(n+1)/(n+3)} |zt|^{(2n+2)/(n+3)}} \\ &\ll \sum_{a, c} \frac{B^{1/(n+1)} \log B}{\max\{|a|, |c|\}^{(n^2+3n)/(n+3)}} \ll B^{\frac{1}{n+1}} (\log B)^2. \end{aligned}$$

We now apply a Möbius inversion to the condition $\gcd(y, w) = 1$, so that, after using the transformation $w \mapsto [\beta, \gamma]w$, we obtain that $16 \cdot N_{U', H'}(B^{\frac{1}{n+1}})$ is equal to

$$\sum_{\substack{|a|, |c| \geq 0 \\ |y|, |z|, |t|, |w| \geq 1 \\ \gcd(a, c) = \gcd(yz, t) = 1}} \sum_{\substack{\alpha, \beta, \gamma > 0 \\ \alpha | z \\ \gamma | y}} \frac{\mu(\alpha)\mu(\beta)\mu(\gamma)}{[\alpha, \beta]} V_{1,1}(a, c, y, z, t, [\beta, \gamma]w; B) + O(B^{\frac{1}{n+1} + \epsilon}).$$

Replacing the sum over w by an integral, we obtain

$$\sum_{|w| \geq 1} V_{1,1}(a, c, y, z, t, [\beta, \gamma]w; B) = \int_{|w| \geq 1} V_{1,1}(a, c, y, z, t, [\beta, \gamma]w; B) dw + O(R_1),$$

where

$$R_1 = \max_w V_{1,1}(a, c, y, z, t, [\beta, \gamma]w; B) \ll \frac{B^{1/(n+3)}}{\max\{|a|, |c|\}^{(n+5)/(n+3)} |y|^{2/(n+3)}}.$$

Using the fact that $\mathcal{H}(0, 0, [\beta, \gamma]w, \dots) \leq B$ and $|w| \geq 1$ imply

$$\begin{aligned} \max\{|a|, |c|\}^{n^2+2n+1} |z|^{n+1} |\beta|^{n+3} |t|^{n^2+3n+2} &\leq B, \\ |y|^{n^2+2n+1} |z|^{n^2+3n+2} |\beta|^{n+3} |t|^{n+1} &\leq B, \end{aligned}$$

we obtain the estimate

$$\begin{aligned} \sum_{a, c, y, z, t} \sum_{\substack{\alpha | z \\ \beta \\ \gamma | y}} \frac{R_1}{[\alpha, \beta]} &\ll \sum_{a, c, y, z, t} \sum_{\beta} \frac{2^{\omega(z)+\omega(y)} B^{1/(n+3)}}{\beta \max\{|a|, |c|\}^{(n+5)/(n+3)} |y|^{2/(n+3)}} \\ &\ll \sum_{z, t} \sum_{\beta} \frac{2^{\omega(z)} B^{(1+2/(n+1))/(n+3)} \log B}{\beta^{(n+3)/(n+1)} |zt|} \\ &\ll B^{\frac{1+2/(n+1)}{n+3}} (\log B)^4 \end{aligned}$$

Hence we obtain that $16 \cdot N_{U', H'}(B^{\frac{1}{n+1}})$ is equal to

$$\sum_{\substack{|a|, |c| \geq 0 \\ |y|, |z|, |t| \geq 1 \\ \gcd(a, c) = 1 \\ \gcd(yz, t) = 1}} \sum_{\substack{\alpha, \beta, \gamma > 0 \\ \alpha | z \\ \gamma | y}} \frac{\mu(\alpha)\mu(\beta)\mu(\gamma)}{[\alpha, \beta][\beta, \gamma]} \int_{|w| \geq [\beta, \gamma]} V_{1,1}(a, c, y, z, t, w; B) dw + O(B^{\frac{1}{n+1} + \epsilon})$$

Removing the condition $|w| \geq [\beta, \gamma]$, we obtain

$$\int_{|w| \geq [\beta, \gamma]} V_{1,1}(a, c, y, z, t, w; B) dw = V_2(a, c, y, z, t; B) + O(R_2),$$

where, using the geometric mean of the conditions

$$\max\{|a|, |c|\}^{\frac{n^2+2n+1}{2}} |y|^{\frac{n^2+2n+1}{2}} |zt|^{\frac{n^2+4n+3}{2}} |w|^{n+3} \leq B,$$

(implied by $\mathcal{H}(0, 0, \dots) \leq B$) with weight $\delta := \frac{2}{n+1} + \epsilon(n+3)$ and of $|w| \leq [\beta, \gamma]$ with weight $1 - \delta$,

$$\begin{aligned} R_2 &= \int_{|w| \leq [\beta, \gamma]} V_{1,1}(a, c, y, z, t, w; B) dw \\ &\ll \frac{[\beta, \gamma]^{1-\delta} B^{1/(n+1)+\epsilon}}{\max\{|a|, |c|\} \max\{|ay|, |cy|\}^{1+\epsilon(n^2+2n+1)/2} |zt|^{1+\epsilon(n^2+4n+3)/2}} \end{aligned}$$

for every sufficiently small $\epsilon > 0$. Summing R_2 over the remaining variables gives the error term

$$\sum_{a,c,y,z,t} \sum_{\substack{\alpha|z \\ \beta \\ \gamma|y}} \frac{|\mu(\alpha)\mu(\beta)\mu(\gamma)|R_2}{[\alpha,\beta][\beta,\gamma]} \ll_{\epsilon} \sum_{\beta} \frac{B^{1/(n+1)+\epsilon}}{\beta^{1+\delta}} \ll_{\epsilon} B^{\frac{1}{n+1}+\epsilon}.$$

Hence $16 \cdot N_{U',H'}(B^{\frac{1}{n+1}})$ is equal to

$$\begin{aligned} & \sum_{\substack{|a|,|c|\geq 0 \\ |y|,|z|,|t|\geq 1 \\ \gcd(a,c)=\gcd(yz,t)=1}} \sum_{\substack{\alpha,\beta,\gamma>0 \\ \alpha|z \\ \gamma|y}} \frac{\mu(\alpha)\mu(\beta)\mu(\gamma)}{[\alpha,\beta][\beta,\gamma]} V_2(a,c,y,z,t;B) + O_{\epsilon}(B^{\frac{1}{n+1}+\epsilon}) \\ = & \sum_{\substack{|a|,|c|\geq 0 \\ |y|,|z|,|t|\geq 1 \\ \gcd(a,c)=\gcd(yz,t)=1}} \sum_{\substack{\alpha,\beta,\gamma>0 \\ \alpha|z \\ \gamma|y}} \frac{\mu(\alpha)\mu(\beta)\mu(\gamma)}{[\alpha,\beta][\beta,\gamma]} V_2(a,c,y,z,t;1) B^{\frac{2}{n+3}} + O_{\epsilon}(B^{\frac{1}{n+1}+\epsilon}), \end{aligned}$$

where we have applied the transformation

$$(b,w) \mapsto B^{\frac{1}{n+3}}(b,w) \text{ or } (d,w) \mapsto B^{\frac{1}{n+3}}(d,w)$$

inside the integral.

Next, we apply the transformation,

$$(at, ct, yz, z, t) \mapsto (\hat{a}, \hat{c}, \hat{y}, z, t)$$

and then

$$(b,w) \mapsto ((zt)^{\frac{2}{n+3}}b, (zt)^{\frac{-n-1}{n+3}}w) \text{ or } (d,w) \mapsto ((zt)^{\frac{2}{n+3}}d, (zt)^{\frac{-n-1}{n+3}}w)$$

inside the integral to obtain that $16 \cdot N_{U',H'}(B^{\frac{1}{n+1}})$ is equal to

$$\begin{aligned} & \sum_{\substack{|\hat{a}|,|\hat{c}|\geq 0 \\ |\hat{y}|,|z|,|t|\geq 1 \\ t|\gcd(\hat{a},\hat{c}) \\ z|\hat{y} \\ \gcd(\hat{a}/t,\hat{c}/t)=1 \\ \gcd(\hat{y},t)=1}} \sum_{\substack{\alpha,\beta,\gamma>0 \\ \alpha|z \\ \gamma|\hat{y}/z}} \frac{\mu(\alpha)\mu(\beta)\mu(\gamma)}{[\alpha,\beta][\beta,\gamma]} V_2(\hat{a}/t, \hat{c}/t, \hat{y}/z, z, t; 1) B^{\frac{2}{n+3}} + O_{\epsilon}(B^{\frac{1}{n+1}+\epsilon}) \\ = & \sum_{\substack{|\hat{a}|,|\hat{c}|\geq 0 \\ |\hat{y}|\geq 1 \\ \gcd(\hat{a},\hat{c},\hat{y})=1 \\ (\hat{a},\hat{c})\neq(0,0)}} \vartheta(\hat{a}, \hat{c}, \hat{y}) B^{\frac{2}{n+3}} + O_{\epsilon}(B^{\frac{1}{n+1}+\epsilon}), \end{aligned}$$

where

$$\begin{aligned} \vartheta(\hat{a}, \hat{c}, \hat{y}) &:= \sum_{\substack{|z|,|t|\geq 1 \\ |t|=\gcd(\hat{a},\hat{c}) \\ z|\hat{y} \\ \alpha,\beta,\gamma>0 \\ \alpha|z \\ \gamma|\hat{y}/z}} \frac{\mu(\alpha)\mu(\beta)\mu(\gamma)}{[\alpha,\beta][\beta,\gamma]} |z|^{\frac{-n+1}{n+3}} |t|^{\frac{4}{n+3}} V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \\ &= 2|\gcd(\hat{a}, \hat{c})|^{\frac{4}{n+3}} \sum_{\substack{|z|\geq 1 \\ z|\hat{y} \\ \alpha,\beta,\gamma>0 \\ \alpha|z \\ \gamma|\hat{y}/z}} \frac{\mu(\alpha)\mu(\beta)\mu(\gamma)}{[\alpha,\beta][\beta,\gamma]} |z|^{\frac{-n+1}{n+3}} V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \\ &= 4\omega_{\infty,(\hat{a}:\hat{c}:\hat{y})} \prod_{p \text{ prime}} \lambda_p \omega_{p,(\hat{a}:\hat{c}:\hat{y})}. \end{aligned}$$

In total,

$$\begin{aligned}
N_{U', H'}(B^{\frac{1}{n+1}}) &= \sum_{\substack{|\hat{a}|, |\hat{c}| \geq 0 \\ |\hat{y}| \geq 1 \\ \gcd(\hat{a}, \hat{c}, \hat{y})=1 \\ (\hat{a}, \hat{c}) \neq (0,0)}} \left(\frac{1}{4} \omega_{\infty, (\hat{a}:\hat{c}:\hat{y})} \prod_{p \text{ prime}} \lambda_p \omega_{p, (\hat{a}:\hat{c}:\hat{y})} \right) B^{\frac{2}{n+3}} + O_{\epsilon}(B^{\frac{1}{n+1}+\epsilon}) \\
&= \sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus (\mathbb{V}(\hat{a}, \hat{c}) \cup \mathbb{V}(\hat{y}))} \left(\frac{1}{2} \omega_{\infty, x} \prod_{p \text{ prime}} \lambda_p \omega_{p, x} \right) B^{\frac{2}{n+3}} + O_{\epsilon}(B^{\frac{1}{n+1}+\epsilon}),
\end{aligned}$$

as predicted in Section 7. \square

Remark 9.2. We have omitted the details of the calculation of $\vartheta(\hat{a}, \hat{c}, \hat{y})$ since, according to [Pey95, Corollaire 6.2.18], Manin's conjecture is true with Peyre's constant for all heights on $\mathbb{P}_{\mathbb{Q}}^1 \cong \phi'^{-1}(x)$ and hence it follows that $\vartheta(\hat{a}, \hat{c}, \hat{y})$ is equal to $2\mathfrak{c}_{(\hat{a}:\hat{c}:\hat{y})}$.

Proposition 9.3. *We have*

$$\sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus (\mathbb{V}(\hat{a}, \hat{c}) \cup \mathbb{V}(\hat{y}))} \mathfrak{c}_x < \infty.$$

Proof. In the case $\hat{a} \neq 0$, the condition $\max |\mathcal{M}'_n(\hat{a}, b, \hat{c}, (b\hat{c} + \hat{y}^n w)/\hat{a}, \hat{y}, 1, 1, w)| \leq 1$ implies

$$(*) \quad |b|^{n+1} |\hat{a}|^{2n+2} |w|^2 \leq 1 \text{ and } |\hat{a}|^{n^2+2n+1} |w|^{n+3} \leq 1,$$

hence we obtain

$$V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \leq \iint_{(*)} \frac{1}{|\hat{a}|} db dw \ll \int_{(*)} \frac{1}{|\hat{a}|^3 |w|^{2/(n+1)}} dw \ll \frac{1}{|\hat{a}|^{3+(n^2-1)/(n+3)}}.$$

Similarly, we obtain

$$V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \leq \frac{1}{|\hat{c}|^{3+(n^2-1)/(n+3)}}.$$

The condition $\max |\mathcal{M}'_n(\hat{a}, b, \hat{c}, (b\hat{c} + \hat{y}^n w)/\hat{a}, \hat{y}, 1, 1, w)| \leq 1$ also implies

$$(**) \quad |b|^{n+3} |\hat{a}|^2 |\hat{y}|^2 \leq 1 \text{ and } |(b\hat{c} + \hat{y}^n w)/\hat{a}|^{n+3} |\hat{a}|^2 |\hat{y}|^2 \leq 1,$$

hence we obtain

$$\begin{aligned}
V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) &\leq \iint_{(**)} \frac{1}{|\hat{a}|} db dw = \int_{\max\{|b|, |w|\}^{n+3} |\hat{a}\hat{y}|^2 \leq 1} \frac{1}{|\hat{y}|^n} db dw \\
&\ll \frac{1}{|\hat{a}|^{4/(n+3)} |\hat{y}|^{n+4/(n+3)}}.
\end{aligned}$$

Similarly, we obtain

$$V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \ll \frac{1}{|\hat{c}|^{4/(n+3)} |\hat{y}|^{n+4/(n+3)}}.$$

Together, we obtain

$$V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \ll \frac{1}{\max\{|\hat{a}|, |\hat{c}|\}^{4/(n+3)} \max\{|\hat{a}|, |\hat{c}|, |\hat{y}|\}^{(n^2+3n+4)/(n+3)}}.$$

There exist $\lambda_1, \lambda_2 > 0$ with

$$\lambda_1 > \frac{2n+2}{n^2+3n+4}, \quad \lambda_2 > \frac{n+3}{n^2+3n+4}, \quad \lambda_1 + \lambda_2 = 1,$$

so that using $\max\{|\hat{a}|, |\hat{c}|, |\hat{y}|\} \leq \max\{|\hat{a}|, |\hat{c}|\}^{\lambda_1} |\hat{y}|^{\lambda_2}$, we obtain that there exists $\mu > 0$ such that

$$V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \ll \frac{1}{\max\{|\hat{a}|, |\hat{c}|\}^{2+\mu} |\hat{y}|^{1+\mu}}.$$

With $\vartheta(\hat{a}, \hat{c}, \hat{y})$ from the proof of Theorem 9.1, we have

$$\sum_{\substack{\hat{a}, \hat{c} \geq 0 \\ \hat{y} \geq 1 \\ (\hat{a}, \hat{c}) \neq (0, 0)}} \vartheta(\hat{a}, \hat{c}, \hat{y}) \ll \sum_{\hat{a}, \hat{c}} \frac{\gcd(\hat{a}, \hat{c})}{\max\{\hat{a}, \hat{c}\}^{2+\mu}} \sum_{\hat{y}} \frac{4^{\omega(\hat{y})} d(\hat{y})}{\hat{y}^{1+\mu}}.$$

Our aim is to show that this sum converges. We have

$$\begin{aligned} \sum_{\hat{a} \leq \hat{c} \leq M} \frac{\gcd(\hat{a}, \hat{c})}{\max\{\hat{a}, \hat{c}\}^{2+\mu}} &= \sum_{\hat{c} \leq M} \frac{1}{\hat{c}^{2+\mu}} \sum_{\hat{a} \leq \hat{c}} \gcd(\hat{a}, \hat{c}) = \sum_{\hat{c} \leq M} \frac{1}{\hat{c}^{1+\mu}} \sum_{d|\hat{c}} \frac{\phi(d)}{d} \\ &\ll \sum_{\hat{c} \leq M} \frac{1}{\hat{c}^{1+\mu}} \sum_{d|\hat{c}} 1 = \sum_{\hat{c} \leq M} \frac{d(\hat{c})}{\hat{c}^{1+\mu}} \\ &\ll \sum_{\hat{c} \leq M} \frac{d(\hat{c})}{M^{1+\mu}} + \int_1^M \sum_{\hat{c} \leq \lambda} \frac{d(\hat{c})}{\lambda^{2+\mu}} d\lambda \\ &\ll \frac{\log M}{M^\mu} + \int_1^M \frac{\log \lambda}{\lambda^{1+\mu}} d\lambda \ll 1. \end{aligned}$$

Note that we have

$$\begin{aligned} \sum_{\hat{y} \leq M} 4^{\omega(\hat{y})} d(\hat{y}) &= \sum_{\hat{y} \leq M} \sum_{z|\hat{y}} 4^{\omega(\hat{y})} = \sum_{\substack{y, z \geq 1 \\ yz \leq M}} 4^{\omega(yz)} \ll \sum_{y \leq M} 4^{\omega(y)} \sum_{z \leq M/y} 4^{\omega(z)} \\ &\ll \sum_{y \leq M} \frac{4^{\omega(y)} M (\log M)^3}{y} \ll M (\log M)^7. \end{aligned}$$

It follows that we have

$$\begin{aligned} \sum_{\hat{y} \leq M} \frac{4^{\omega(\hat{y})} d(\hat{y})}{\hat{y}^{1+\mu}} &\ll \sum_{\hat{y} \leq M} \frac{4^{\omega(\hat{y})} d(\hat{y})}{M^{1+\mu}} + \int_1^M \sum_{\hat{y} \leq \lambda} \frac{4^{\omega(\hat{y})} d(\hat{y})}{\lambda^{2+\mu}} d\lambda \\ &\ll \frac{(\log M)^7}{M^\mu} + \int_1^M \frac{(\log M)^7}{M^{1+\mu}} d\lambda \ll 1. \end{aligned} \quad \square$$

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